K22P 1408

Reg. No. : ....

Name : ....

III Semester M.Sc. Degree (CBSS – Reg./Sup./Imp.) Examination, October 2022 (2019 Admission Onwards) MATHEMATICS MAT3C11 : Number Theory

Time : 3 Hours | Max. Marks : 80

PART – A

Answer **any four** questions from Part **A**. **Each** question carries **4** marks.

1. Prove that the infinite series $\sum_{n=1}^{\infty} 1/P_n$ diverges.
2. State and prove Euclid's lemma.
3. If f is multiplicative then prove that f(1) = 1.
4. Assume that (a, m) = d. Then prove that the linear congruence $ax \equiv b$ (mod) m has solutions if and only if d|b.
5. Determine whether 219 is a quadratic residue or non residue mod 383.
6. Prove that an algebraic number α is an algebraic integer if and only if its minimum polynomial over Q has coefficients in Z.

PART – B

Answer **any four** questions from Part **B not** omitting **any** Unit, **Each** question carries **16** marks.

Unit – 1

7. a) State and prove the division algorithm.
   b) Prove that every integer n > 1 is either a prime number or a product of prime numbers.

P.T.O.

8. a) If $n \geq 1$, then Prove that $\phi(n) = \sum_{d|n} \mu(d) \frac{n}{d}$.

   b) Assume f is multiplicative. Prove that $f^{-1}(n) = \mu(n) f(n)$ for every square free n.

9. a) State and prove Lagrange's theorem.

   b) Solve the congruence $5x \equiv 3 \pmod{24}$.

Unit – 2

10. a) Prove that the Legendre' symbol $(n|p)$ is a completely multiplicative function of n.

    b) State and prove quadratic reciprocity law.

11. a) Let $(a, m) = 1$. Then prove that if a is a primitive root mod m if and only if the numbers $a, a^2, ..., a^{\phi(m)}$ form a reduced residue system mod m.

    b) If p is an odd prime and $\alpha \geq 1$ then prove that there exist an odd primitive roots g modulo p^{α} and each such g is also a primitive root modulo $2p^{\alpha}$.

12. a) Write in detail any one application of primitive roots in cryptography.

    b) Solve the superincreasing knapsack problem.

    $28 = 3x_1 + 5x_2 + 11x_3 + 20x_4 + 41x_5$

Unit – 3

13. a) Prove that every subgroup H of a free abelian group G of rank n is free of rank $s \leq n$. Moreover there exist a basis $u_1, u_2, ..., u_n$ of G and positive integers $\alpha_1, \alpha_2, ..., \alpha_s$ such that, $\alpha_1 u_1, \alpha_2 u_2, ..., \alpha_s u_s$ is a basis for H.

    b) Let G be a free abelian group of rank n with basis $\{x_1, x_2, ..., x_n\}$. Suppose (a_{ij}) is an $n \times n$ matrix with integer entries. Then prove that the elements $y_i = \sum_j a_{ij} x_j$ form a basis of G if and only if (a_{ij}) is unimodular.

14. a) Suppose $\{\alpha_1, \alpha_2, ..., \alpha_n\} \in D$ form a Q-basis for K. Then prove that if $\Delta[\alpha_1, \alpha_2, ..., \alpha_n]$ is square free then $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ is an integral basis.

b) Prove that every number field K possess an integral basis and the additive group of D is free abelian group of rank n equal to the degree of K.

15. a) Let d be a square free rational integer. Then prove that the integers of $Q(\sqrt{d})$ are

a) $Z|\sqrt{d}|$ if $d \not\equiv 1 \pmod 4$

b) $Z\left|\frac{1}{2}+\frac{1}{2}\sqrt{d}\right|$ if $d \not\equiv 1 \pmod 4$.

b) Prove that the minimum polynomial of $\xi = e^{\frac{2\pi i}{p}}$, p an odd prime, over Q is $f(t) = t^{p-1} + t^{p-2} + ... + t + 1$ and the degree of $Q(\xi)$ is $p - 1$.

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III Semester M.Sc. Degree (CBSS – Reg./Supple./Imp.) Examination, October 2023
(2020 Admission Onwards)
MATHEMATICS
MAT3C11 : Number Theory

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any four** questions from Part **A**. **Each** question carries **4** marks.

1. Prove that if (a, b) = 1 then $(a^n, b^k) = 1$ for all $n \geq 1$, $k \geq 1$.

2. Find all integers such that $\phi(n) = \frac{n}{2}$.

3. Find the quadratic residues and non residue modulo 11.

4. Encrypt the message “RETURN HOME” using caeser ciphar.

5. Define an R-module. Find all submodules of $\mathbb{Z}$-module.

6. Check whether $e^{\frac{2\pi i}{23}}$ is algebraic integer or not ?

PART – B

Answer **any four** questions from Part **B** not omitting **any** Unit. **Each** question carries **16** marks.

Unit – 1

7. a) State and prove fundamental theorem of arithmetic.

   b) Given that a and b are integers with b > 0. Then prove that there exists a unique pair of integers q and r such that a = bq + r, with $0 \leq r < b$ and r = 0 if and only if b|a.

8. a) If $n \geq 1$, prove that $\sum_{d|n} \phi(d) = n$.

   b) Assume f is multiplicative. Prove that f is completely multiplicative if and only if $f^{-1}(n) = \mu(n) f(n)$ for all $n \geq 1$.

P.T.O.

9. a) State and prove Chinese remainder theorem.

   b) Find all positive integers n for which $n^{13} \equiv n \pmod{1365}$.

Unit – 2

10. a) State and prove Gauss' lemma.

    b) Define Jacobi symbol and prove that $(-1/p) = (-1)^{\frac{p-1}{2}}$ and $(2/p) = (-1)^{\frac{p^2-1}{8}}$.

11. a) Suppose (a, m) = 1. Prove that a is a primitive root modulo m if and only if the numbers a, a^2, ..., $a^{\phi(m)}$ form a reduced residue system modulo m.

    b) If p is an odd prime and $\alpha \le 1$ then prove that there exist odd primitive roots g modulo p^α and each such g is also a primitive root modulo $2p^\alpha$.

12. a) Explain RSA public key algorithm with an example.

    b) Obtain all solutions of the knapsack problem
    $28 = 3x_1 + 5x_2 + 11x_3 + 20x_4 + 41x_5$.

Unit – 3

13. a) Given R is a ring. Then prove that every symmetric polynomial in $R[t_1, \ldots, t_n]$ is expressible as a polynomial with coefficients in R in the elementary symmetric polynomials $s_1, \ldots, s_n$.

    b) Let G be a free abelian group of rank r and H is a subgroup of G. Then prove that G/H is finite if and only if the rank of G and H are equal.

14. a) Prove that the set A of algebraic numbers is a subfield of the complex field $\mathbb{C}$.

    b) Prove that a complex number θ is an algebraic integer if and only if the additive group generated by all powers 1, θ, θ^2, ... is finitely generated.

15. a) If d is a square-free rational integer, then prove that the integers of $\mathbb{Q}(\sqrt{d})$ are

    $\mathbb{Z}\left[\sqrt{d}\right]$ if $d \not\equiv 1 \pmod 4$

    $\mathbb{Z}\left[\frac{1}{2} + \frac{1}{2}\sqrt{d}\right]$ if $d \equiv 1 \pmod 4$

    b) Prove that the ring $\mathfrak{D}$ of integers $\mathbb{Q}(\zeta)$ is $\mathbb{Z}[\zeta]$.

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III Semester M.Sc. Degree (C.B.S.S. – Reg./Supple./Imp.) Examination, October 2023
(2020 Admission Onwards)
MATHEMATICS
MAT3C12 : Functional Analysis

Time : 3 Hours Max. Marks : 80

PART – A

Answer **four** questions from this Part. **Each** question carries **4** marks.

1. State and prove Riesz lemma.

2. Show that c_{00} cannot be a Banach space with respect to any norm.

3. If a closed map F is bijective, then show that its inverse F^{-1} is also closed.

4. State open mapping theorem.

5. Let X be an inner product space and $x \in X$. Prove that $\langle x, y \rangle = 0$ for all $y \in X$ if and only if x = 0.

6. Let E be an orthogonal subset of an inner product space X and $0 \notin E$. Show that E is linearly independent.

PART – B

Answer **four** questions from this Part without omitting any Unit. **Each** question carries **16** marks.

Unit – I

7. a) Define a normed space and draw the sets $\{x \in \mathbb{R}^2 ; \| x \|_p = 1\}$ for p = 1, 2 and ∞.

   b) If X is a finite dimensional normed space then show that every closed and bounded subset of X is compact.

8. a) Show that every linear map from a finite dimensional normed space is continuous.

   b) Let X and Y be normed spaces and $F : X \to Y$ be a linear map such that R(F) of F is finite dimensional. Show that F is continuous if and only if the zero space Z(F) is closed in X.

9. a) State and prove Hahn-Banach separation theorem.

   b) If X is a normed space and X' is strictly convex then show that for every subspace Y of X and every $g \in Y'$, there is a unique Hahn-Banach extension of g to X.

Unit – II

10. a) State and prove Uniform Boundedness Principle.

    b) Give the geometric interpretation of Uniform Boundedness Principle.

11. State and prove Closed Graph Theorem.

12. a) State and prove Bounded Inverse Theorem.

    b) Let X be a Banach space in the norm $\| \ \|$. Show that there is a norm $\| \ \|'$ on X which is comparable to the norm $\| \ \|$, but in which X is not complete.

Unit – III

13. a) State and prove Gram-Schmidt orthonormalization process.

    b) State and prove Riesz-Fischer theorem.

14. a) If H is a non-zero separable Hilbert space over K then show that H has a countable orthonormal basis.

    b) If E is a convex subset of an inner product space X, then show that there exists at most one best approximation from E to X.

15. a) State and prove Riesz representation theorem.

    b) Let H be a Hilbert space and for $f \in H'$, let y_f be the representer of f in H. Show that the map $T : H \to H'$ given by $T(f) = y_f$ is a surjective conjugate-linear isometry.

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III Semester M.Sc. Degree (CBSS – Reg./Sup./Imp.)
Examination, October 2022
(2019 Admission Onwards)
MATHEMATICS
MAT 3C13 : Complex Function Theory

Time : 3 Hours Max. Marks : 80

PART – A

Attempt **any four** questions from this Part. **Each** question carries **4** marks.

1. Define the following terms :

   i) Period module of a meromorphic function

   ii) Discrete module.

2. Show that the series $\sum_{n=1}^{\infty} n^{-z}$ converges uniformly and absolutely on a subset of the complex plane $\mathbb{C}$.

3. Is $\mathbb{C} - \{0\}$ is simply connected ? Justify your answer.

4. Is the sets $\{z : |z| < 1\}$ and $\mathbb{C}$ are homeomorphic ? Justify your answer.

5. Prove that a harmonic function u in $\mathbb{C}$ is infinitely differentiable.

6. Given that v_1 and v_2 are two harmonic conjugates of a harmonic function u. Prove that $v_2 - v_1 = c$, where c is a constant.

P.T.O.

PART – B

Answer **any four** questions from this Part without omitting any Unit. **Each** question carries **16** marks.

Unit – I

7. a) Prove the following :

i) Let $S = \{z : \text{Re}z \geq a\}$ where $a > 1$. If $\varepsilon > 0$, then there is a number $\delta > 0$, $0 < \delta < 1$, such that for all $z \in S$, $\left|\int_{\alpha}^{\beta} (e^t - 1)^{-1} t^{z-1} dt\right| < \varepsilon$ whenever $\delta > \beta > \alpha$.

ii) Let $S = \{z : \text{Re}z \leq A\}$ where $-\infty < A < \infty$. If $\varepsilon > 0$, then there is a number $k > 1$ such that for all $z \in S$, $\left|\int_{\alpha}^{\beta} (e^t - 1)^{-1} t^{z-1} dt\right| < \varepsilon$ whenever $\beta > \alpha > k$.

b) Prove : A non-constant elliptic function has equally many poles as it has zeroes.

8. With the usual notations, prove that :

a) $\wp(2z) = \frac{1}{4}\left(\frac{\wp''(z)}{\wp'(z)}\right)^2 - 2\wp(z)$

b) $\wp'(z) = -\sigma(2z)/\sigma(z)^4$

c) $\begin{vmatrix} \wp(z) & \wp'(z) & 1 \\ \wp(u) & \wp'(u) & 1 \\ \wp(u+z) & -\wp'(u+z) & 1 \end{vmatrix} = 0$

d) $\frac{\wp'(z)}{\wp(z) - \wp(u)} = \zeta(z-u) + \zeta(z+u) - 2\zeta(z)$

9. a) Prove that Riemann's zeta function ζ has no other zeroes outside the closed strip $\{z : 0 \leq z \leq 1\}$.

b) Prove that if Re $z > 1$, then $\zeta(z) = \prod_{n=1}^{\infty}\left(\frac{1}{1 - p_n^{-z}}\right)$ where p_n is a sequence of prime numbers.

Unit – II

10. State and prove Schwarz Reflection Principle.

11. a) Let $\gamma : [0, 1] \to \mathbb{C}$ be a path and let $\{(f_t, D_t) : 0 \leq t \leq 1\}$ be an analytic continuation along γ. Show that $\left\{\left(f_t', D_t\right) : 0 \leq t \leq 1\right\}$ is also a continuation along γ.

    b) Let (f, D) be a function element which admits unrestricted continuation in the simply connected region G. Prove that there is an analytic function $F : G \to \mathbb{C}$ such that $F(z) = f(z)$ for all z in D.

    c) Is the region $\{z \in \mathbb{C} : 1 < |z| < 2\}$ is simply connected ? Justify your answer.

12. State and prove the Mittag-Leffler's theorem.

Unit – III

13. a) State and prove Jensen's formula.

    b) State and prove Maximum Principle (Second Version).

14. Prove that the Dirchlet problem can be solved in a unit disk.

15. a) Define the Poisson kernel $P_r(\theta)$. Prove that $P_r(\theta) = \text{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right)$.

    b) Prove that $P_r(\theta) < P_r(\delta)$ if $0 < \delta < |\theta| \leq \pi$.

    c) For $|z| < 1$ let $u(z) = \text{Im}\left[\left(\frac{1+z}{1-z}\right)^2\right]$. Show that u is harmonic.

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III Semester M.Sc. Degree (C.B.S.S. – Reg./Supple./Imp.)
Examination, October 2023
(2020 Admission Onwards)
MATHEMATICS
MAT3C13 : Complex Function Theory

Time : 3 Hours   Max. Marks : 80

PART – A

Answer **any four** questions. **Each** question carries **4** marks.

1. Prove that the sum of the residues of an elliptic function is zero.

2. Define the period module. Show that if f is not a constant function, then the elements of the period module of f are isolated.

3. Let $\gamma : [0,1] \to \mathbb{C}$ be a path from a to b and let $\{(f_t, D_t) : 0 \le t \le 1\}$ and $\{(g_t, B_t) : 0 \le t \le 1\}$ be analytic continuations along γ such that $[f_0]_a = [g_0]_a$. Prove that $[f_1]_b = [g_1]_b$.

4. Show that if G an open connected subset of $\mathbb{C}$, is homeomorphic to the unit disk, then G is simply connected.

5. a) Prove that if $u : G \to \mathbb{C}$ is harmonic, then u is infinitely differentiable.

   b) Define the mean value property.

6. Prove that if $u : G \to \mathbb{R}$ is a continuous function which has the MVP, then u is harmonic.

P.T.O.

PART – B

Answer **any four** questions without omitting **any** Unit. **Each** question carries **16** marks.

Unit – I

7. a) Define basis of a period module. Prove that any two bases of the same module are connected by a unimodular transformation.

   b) Prove that an elliptic function without poles is a constant.

8. a) Prove that a non-constant elliptic function has equally many poles as it has zeros.

   b) Prove that zeros $a_1, a_2, ..., a_n$ and poles $b_1, b_2, ... ,b_n$ of an elliptic function satisfy $a_1 + a_2 + ... + a_n \equiv b_1 + b_2 + ... + b_n \pmod{M}$.

9. a) Prove that for Rez > 1, $\zeta(z)\,\Gamma(z) = \int_0^\infty (e^t - 1)^{-1} t^{z-1} dt$.

   b) Define Riemann's functional equation. State and prove Euler's theorem.

Unit – II

10. State and prove Runge's theorem.

11. State and prove Mittag-Leffler's theorem.

12. a) When does a function element (f,D) said to admit unrestricted analytic continuation in G ?

    b) State and prove Monodromy theorem.

Unit – III

13. a) State and prove Jensen's formula. Also state Poisson-Jensen formula.

    b) Suppose f(0) $\neq$ 0 in Jensen's formula. Show that if f has a zero at z = 0 of multiplicity m, then $\log\left|\frac{f^{(m)}(0)}{m!}\right| + m\log r = -\sum_{k=1}^{n}\log\left(\frac{r}{|a_k|}\right) + \frac{1}{2\pi}\int_0^{2\pi}\log\left|f(re^{i\theta})\right|d\theta$.

14. a) Define subharmonic and superharmonic function. When does one say that a function satisfies the maximum principle ?

b) Let G be a region and $\phi : G \to \mathbb{R}$ be a continuous function. Then prove that ϕ is subharmonic iff for every region G_1 contained in G and every harmonic function u_1 on G_1, $\phi - u_1$ satisfies the maximum principle on G_1.

c) If ϕ_1 and ϕ_2 are subharmonic functions on G and if $\phi(z) = \max\{\phi_1(z), \phi_2(z)\}$ for each z in G, then show that ϕ is a subharmonic function.

15. Let $D = \{z : |z| < 1\}$ and suppose that $f : \partial D \to \mathbb{R}$ is a continuous function. Then prove that there is a continuous function $u : \bar{D} \to \mathbb{R}$ such that

a) $u(z) = f(z)$ for z in ∂D.

b) u is harmonic in D. Also show u is unique and is defined by the formula

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\, f(e^{it})\, dt, \text{ for } 0 \le r < 1, 0 \le \theta \le 2\pi.$$

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III Semester M.Sc. Degree (CBSS – Reg./Sup./Imp.) Examination, October 2022
(2019 Admission Onwards)
MATHEMATICS
MAT3C14 – Advanced Real Analysis

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any four** questions from this Part. **Each** question carries **4** marks. **(4×4=16)**

1. Let B be the uniform closure of an algebra A of bounded functions. Then prove that B is a uniformly closed algebra.

2. Give an example of a functions with f_n converges to f, but f_n' does not converges to f'. Justify your answer.

3. Define orthogonal system of functions. Give example with justification.

4. Prove that $\lim_{x \to +\infty} x^{-\alpha} \log x = 0$.

5. Prove that the existence of all partial derivatives does not imply the differentiability.

6. Explain directional derivative of f at x in the direction of a unit vector u and continuously differentiable functions.

PART – B

Answer **any four** questions from this Part without omitting any Unit. **Each** question carries **16** marks. **(4×16=64)**

Unit – I

7. a) Suppose $f_n \to f$ uniformly on a set E in a metric space. Let x be a limit point of E, and suppose that $\lim_{t \to x} f_n(t) = A_n$, (n = 1, 2, 3, ...). Then Prove that $\{A_n\}$ converges and $\lim_{t \to x} f(t) = \lim_{t \to \infty} A_n$.

P.T.O.

b) Suppose K is compact, and

i) $\{f_n\}$ is a sequence of continuous functions on K,

ii) $\{f_n\}$ converges pointwise to a continuous function f on K,

iii) $f_n(x) \geq f_{n+1}(x)$ for all $x \in K$, n = 1, 2, 3 ... Then prove that $f_n \to f$ uniformly on K.

8. a) Prove that there exists a real continuous function on the real line which is nowhere differentiable.

b) Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

9. Let A be an algebra of real continuous functions on a compact set K. If A separates points on K and if A varnishes at no point of K, then prove that the uniform closure B of A consists of all real continuous functions on K.

Unit – II

10. a) Suppose the series $\sum_{n=0}^{\infty} c_n x^n$ converges for $|x| < R$ and define $f(x) = \sum_{n=0}^{\infty} c_n x^n$, $(|x| < R)$. Then prove that the series $\sum_{n=0}^{\infty} c_n x^n$ converges uniformly on $[-R+\in, R-\in]$, no matter which $\in > 0$ is chosen. Also prove that the function f is continuous and differentiable in (– R, R) and $f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$, $|x| < R$.

b) Suppose the series $\sum_{n=0}^{\infty} c_n x^n$ converges for $|x| < R$ and define $f(x) = \sum_{n=0}^{\infty} c_n x^n$, $(|x| < R)$. Then prove that f has derivatives of all orders in (– R, R) and derive the formulas.

11. State and prove Parseval's Theorem.

12. a) Define Gamma Function. Prove that $\log\Gamma$ is convex on $(0, \infty)$.

b) State and prove Stiriling's Formula.

Unit – III

13. a) Let r be a positive integer. If a vector space X is spanned by a set of r vectors, then prove that $\dim X \leq r$.

b) Suppose X is a vector space, and dim X = n. Prove that

i) A set E of n vectors in X spans X if and only if E is independent.

ii) X has a basis and every basis consists of n vectors.

iii) If $1 \leq r \leq n$ and $\{y_1, y_2, ..., y_r\}$ is an independent set in X then X has a basis containing $\{y_1, y_2, ..., y_r\}$.

14. a) Suppose f maps an open set $E \subset R^n$ into R^m. Then prove that $f \in C(E)$ if and only if the partial derivatives $D_j f_i$ exist and are continuous on E for $1 \leq i \leq m$, $1 \leq j \leq n$.

b) Suppose f maps a convex open set $E \subset R^n$ into R^m, f is differentiable in E and there is a real number M such that $\|f'(x)\| \leq M$ for every $x \in E$. Then prove that $|f(b) - f(a)| \leq M|b - a|$ for all $a \in E$, $b \in E$.

15. State and prove implicit function theorem.

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III Semester M.Sc. Degree (C.B.S.S. – Reg./Supple./Imp.)
Examination, October 2023
(2020 Admission Onwards)
MATHEMATICS
MAT3C14 : Advanced Real Analysis

Time : 3 Hours | Max. Marks : 80

PART – A

Answer **four** questions from this Part. **Each** question carries **4** marks.

1. Distinguish between pointwise boundedness and uniform boundedness of sequence of functions on a set E.

2. Define the limit function of sequence $\{f_n\}$ of functions and show that for $m, n = 1, 2, 3, \ldots$, if $S_{m,n} = \frac{m}{m+n}$, then $\lim_{n\to\infty} \lim_{m\to\infty} S_{m,n} \neq \lim_{m\to\infty} \lim_{n\to\infty} S_{m,n}$.

3. Define beta function.

4. Show that the functional equation $\Gamma(x + 1) = x\Gamma(x)$ holds if $0 < x < \infty$.

5. Prove that a linear operator A on a finite-dimensional vector space X is one-to-one if and only if the range of A is all of X.

6. State the implicit function theorem. **(4×4=16)**

PART – B

Answer **4** questions from this Part without omitting **any** Unit. **Each** question carries **16** marks.

Unit – I

7. State and prove the Stone-Weierstrass theorem.

8. a) Show that there exists a real continuous function on the real line which is nowhere differentiable.

   b) If $\{f_n\}$ is a pointwise bounded sequence of complex functions on a countable set E, then show that the $\{f_n\}$ has a subsequence $\{f_{nk}\}$ such that $\{f_{n_k}(x)\}$ converges for every $x \in E$.

P.T.O.

9. a) If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E, then prove that $\{f_n + g_n\}$ converges uniformly on E.

   b) If $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, then prove that $\{f_n \cdot g_n\}$ converges uniformly on E.

   c) Suppose $\{f_n\}$ is a sequence of functions defined on E, and suppose $|f_n(x)| \leq M_n$ for $x \in E$ and n = 1, 2, 3, ..., then prove that $\sum f_n$ converges uniformly on E if $\sum M_n$ converges.

Unit – II

10. a) Suppose that the series $\sum_{n=0}^{\infty} c_n x^n$ converges for $|x| < R$, and if $f(x) = \sum_{n=0}^{\infty} c_n x^n$, then prove that the function f is continuous and differentiable in $(-R, R)$, and $f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$ where $|x| < R$.

    b) State and prove Taylor's theorem.

11. State and prove Parseval's theorem.

12. a) If $x > 0$ and $y > 0$, then show that $\int_0^1 t^{x-1}(1-t)^{y-1}\, dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.

    b) If f is continuous (with period 2π) and if $\varepsilon > 0$, then prove that there is a trigonometric polynomial P such that $|P(x) - f(x)| < \varepsilon$ for all real x.

Unit – III

13. a) Define dimension of a vector space.

    b) Let r be a positive integer, if a vector space is spanned by a set of r vectors, then prove that $\dim X \leq r$.

    c) Show that $\dim \mathbb{R}^n = n$.

14. a) Define a continuously differentiable mapping.

    b) Suppose f maps an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$. Then prove that $f \in \mathscr{C}'(E)$ if and only if the partial derivatives $D_j f_i$ exist and are continuous on E for $1 \leq i \leq m$, $1 \leq j \leq m$.

15. State and prove inverse function theorem. **(4×16=64)**

K22P 1412

Reg. No. : ....

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III Semester M.Sc. Degree (CBSS – Reg./Sup./Imp.) Examination, October 2022
(2019 Admission Onwards)
MATHEMATICS
MAT 3 E01 : Graph Theory

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any four** questions from this Part. **Each** question carries **4** marks. **(4×4=16)**

1. Explain the personal assignment problem.

2. Prove that $\alpha+\beta = \nu$.

3. If $\delta > 0$, then prove that $\alpha' + \beta' = \nu$.

4. Show that the Petersen graph is 4-edge-chromatic.

5. Show that $K_5 - e$ is planar for any edge e of K_5.

6. Let u and v be two distinct vertices of the graph G. Then prove that a set S of vertices of G is u-v separating if and only if every u-v path has at least one internal vertex belonging to S.

PART – B

Answer **any four** questions from this Part without omitting **any** Unit. **Each** question carries **16** marks. **(4×16=64)**

UNIT – I

7. a) Prove that if a simple graph G contains no K_{m+1}, then G is degree majorised by some complete m-partite graph H. Also prove that, if G has the same degree sequence as H, then $G \approx H$.

   b) Show that a connected α-critical graph has no cut vertices.

P.T.O.

8. a) For any graph G, prove that $\chi \leq \Delta + 1$.

   b) If G is a connected simple graph and is neither an odd cycle nor a complete graph, then prove that $\chi \leq \Delta$.

9. a) If G is simple, then prove that $\pi_k(G) = \pi_k(G - e) - \pi_k(G.e)$ for any edge e of G.

   b) State and prove Dirac theorem on k-critical graphs.

UNIT – II

10. If G is simple, then prove that either $\chi' = \Delta$ or $\chi' = \Delta + 1$.

11. a) Prove that, inner (outer) bridges avoid one another.

    b) Prove that an inner bridge that avoids every outer bridge is transferable.

12. Prove that the following three statements are equivalent :

    a) every planar graph is 4-vertex-colourable;

    b) every plane graph is 4-face-colourable;

    c) every simple 2-edge-connected 3-regular planar graph is 3-edge-colourable.

UNIT – III

13. State and prove Menger's theorem.

14. a) Let G be a bipartite graph with bipartition (X, Y). Then prove that G contains a matching that saturates every vertex in X if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$.

    b) If G is a k-regular bipartite graph with k > 0, then G has a perfect matching.

15. a) Prove that every 3-regular graph without out edges has a perfect matching.

    b) Let l be a feasible vertex labelling of G. If G_l contains a perfect matching M*, then prove that M* is an optimal matching of G.

K23P 1412

Reg. No. : ....

Name : ....

III Semester M.Sc. Degree (CBSS – Reg./Supple./Imp.) Examination, October 2023
(2020 Admission Onwards)
MATHEMATICS
MAT3E01 : Graph Theory

Time : 3 Hours | Max. Marks : 80

PART – A

Answer **any 4** questions. **Each** question carries **4** marks. **(4×4=16)**

1. Define independent set of a graph G. Prove that a set $S \subset V$ is an independent set of G if and only if $S - V$ is a covering of G.

2. If $\delta > 0$, then prove that $\alpha' + \beta' = v$ where α' and β' where α' (G) and β' (G) are the edge independence number and edge covering number of G respectively.

3. Show that the Peterson graph is 4–edge chromatic.

4. Prove that a graph G is embeddable in the plane if and only if it is embeddable on the sphere.

5. Prove that if G is a k-regular bipartite graph with k > 0, then G has a perfect matching.

6. Prove that a simple graph G is connected if and only if, given any pair of distinct vertices u and v of G, there are at least n internally disjoint paths from u to v.

P.T.O.

PART – B

Answer **any 4** questions without omitting any **unit**. **Each** question carries **16** marks.

UNIT – I

7. a) State and prove Ramsey's theorem.

   b) Let (S_1, S_2,...,S_n) be any partition of the set of integers 1, 2, ..., r_n. Then, prove that for some i, S_i contains three integers x, y and z satisfying the equation $x + y = z$.

8. a) If $\{x_1, x_2, ..., x_n\}$ is a set of diameter 1 in the plane, then prove that the maximum possible number of pairs of points at distance greater than $1/\sqrt{2}$ is $[n^2/3]$. Also prove that for each n, there is a set $\{x_1, x_2, ..., x_n\}$ of diameter 1 with exactly $[n^2/3]$ pairs of points at distance greater than $1/\sqrt{2}$.

   b) If G is simple and contains no K_{m+1}, then prove that $\varepsilon(G) \leq \varepsilon(T_{m,v})$. Also prove that $\varepsilon(G) = \varepsilon(T_{m,v})$ only if $G = T_{m,v}$.

9. a) If G is k-critical, then prove that $\delta \geq k - 1$.

   b) Show that every k-chromatic graph has at least k vertices of degree at least $k - 1$.

   c) Prove that in a critical graph, no vertex is a clique.

UNIT – II

10. a) If two bridges overlap, then show that either they are skew or else they are equivalent 3-bridges.

    b) Show that $K_{3,3}$ is non-planar.

    c) Prove that an inner bridge that avoids every outer bridge is transferable.

11. a) Let G be a connected graph that is not an odd cycle. Then prove that G has a 2-edge colouring in which both colors are represented at each vertex of degree at least two.

    b) If G is bipartite, then prove that $X' = \Delta$.

12. a) Let M and N be disjoint matchings of G with $|M| > |N|$. Prove that there are disjoint matchings M' and N' of G such that $|M'| = |M| - 1$, $|N'| = |N| + 1$ and $M' \cup N' = M \cup N$.

b) Show that a graph is planar if and only if each of its blocks is planar.

UNIT – III

13. a) Prove that a matching M in G is a maximum matching if and only if G contains no M-augmenting path.

b) In a bipartite graph, show that the number of edges in a maximum matching is equal to the number of vertices in a minimum covering.

14. Prove that G has a perfect matching if and only if $o(G - S) \leq |S|$ for all $S \subset V$.

15. State and prove Menger's theorem. **(4×16=64)**