K21P 4209

Reg. No. : ....

Name : ....

I Semester M.Sc. Degree (C.B.S.S. – Reg./Supple./Imp.) Examination, October 2021
(2018 Admission Onwards)
MATHEMATICS
MAT1C01 : Basic Abstract Algebra

Time : 3 Hours   Max. Marks : 80

PART – A

Answer **four** questions from this Part. **Each** question carries **four** marks.

1. Prove or disprove "the group $\mathbb{Z}_3 \times \mathbb{Z}_3$ is cyclic".

2. Let X be a G-set. Prove that G_x is a subgroup of G for each $x \in X$.

3. Prove that no group of order 30 is simple.

4. Is {(2, 1), (4, 1)} a basis for $\mathbb{Z} \times \mathbb{Z}$? Prove your assertion.

5. Write all polynomials of degree ≤ 3 in $\mathbb{Z}_3[x]$. How many of them are reducible over $\mathbb{Z}_3$?

6. Prove that the p^{th} cyclotomic polynomial is irreducible over $\mathbb{Q}$ for any prime p.

PART – B

Answer **4** questions from this Part without omitting any Unit. **Each** question carries **16** marks.

Unit – I

7. a) Prove that the group $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic and is isomorphic to $\mathbb{Z}_{mn}$ if and only if m and n are relatively prime.

   b) If m is a square free integer then prove that every abelian group of order m is cyclic.

   c) Write all abelian groups of order 32.

P.T.O.

8. a) Let X be a G-set and $x \in X$. Prove that $|Gx| = (G : G_x)$. Also show that if $|G|$ is finite, then $|Gx|$ is a divisor of $|G|$.

   b) Let X be a G-set and $Y \subseteq X$ and $G_Y = \{g \in G | gy = y \text{ for all } y \in Y\}$. Show that G_Y is a subgroup of G.

9. a) State and prove First Sylow Theorem.

   b) Prove that every group of order p^2, where p is a prime, is abelian.

Unit – II

10. Prove that any integral domain D can be enlarged to a field F such that every element of F can be expressed as a quotient of two elements of D.

11. a) Prove that two subnormal (or normal) series of a group G have isomorphic refinements.

    b) Write all composition series of $\mathbb{Z}_{48}$.

12. a) Let $G \neq \{0\}$ be a free abelian group with finite basis. Prove that every bases of G is finite and all basis of G have the same number of elements.

    b) Show that $\mathbb{Q}$ under addition is not a free abelian group.

Unit – III

13. a) Let F be a subfield of a field E and α be any element of E. Prove that the map $\phi_\alpha : F[x] \to E$, defined by $\phi_\alpha (a_0 + a_1x + ... + a_nx^n) = a_0 + a_1\alpha + ... + a_n\alpha^n$ is a homomorphism and $\phi_{\alpha|F}$ is the identity map.

    b) Prove that every nonzero polynomial $f(x) \in F[x]$ of degree n can have at most n zeros in a field F.

14. a) State and prove Eisenstein Criterion.

    b) Let ϕ be a homomorphism of a ring R with unity onto a nonzero ring R′. Let u be a unit in R. Prove that $\phi(u)$ is also a unit in R′.

15. a) Let R be a commutative ring with unity. Prove that M is a maximal ideal of R if and only if R/M is a field.

    b) If F is a field, prove that every ideal in F(x) is principal.

K22P 1601

Reg. No. : ....

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I Semester M.Sc. Degree (CBSS – Reg./Sup./Imp.) Examination, October 2022
(2019 Admission Onwards)
MATHEMATICS
MAT1C01 : Basic Abstract Algebra

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any four** questions from this Part. **Each** question carries **4** marks.

1. List the elements of $\mathbb{Z}_2 \times \mathbb{Z}_4$. Find the order of each the elements.

2. Let X be a G-set. For $x_1, x_2 \in X$, let $x_1 \sim x_2$ if and only if there exists $g \in G$ such that $gx_1 = x_2$. Prove that ~ is an equivalence relation on X.

3. Let N be a normal subgroup of G and H be any subgroup of G. Prove that $H \vee N = HN = NH$.

4. Let H*, H and K be subgroups of G with H* normal in H. Show that $H^* \cap K$ is normal in $H \cap K$.

5. Let $f(x) = 2x^2 + 3x + 4$, $g(x) = 3x^2 + 2x + 3$ in $\mathbb{Z}_6[x]$. Find f(x) + g(x) and f(x)g(x).

6. Let R be a ring with unity 1. Prove that the map $\phi : \mathbb{Z} \to R$ given by $\phi(n) = n \cdot 1$ for $n \in \mathbb{Z}$ is a homomorphism of $\mathbb{Z}$ into R.

PART – B

Answer **any four** questions from this Part without omitting **any** Unit. **Each** question carries **16** marks.

Unit – I

7. a) Let $G_1, G_2, ..., G_n$ be groups. For $(a_1, a_2, ..., a_n)$ and $(b_1, b_2, ..., b_n)$ in $\prod_{i=1}^{n} G_i$, define $(a_1, a_2, ..., a_n)(b_1, b_2, ..., b_n)$ to be the element $(a_1b_1, a_2b_2, ..., a_nb_n)$. Prove that $\prod_{i=1}^{n} G_i$ is a group under this operation.

   b) State Fundamental theorem of finitely generated Abelian groups.

   c) Find all abelian groups of order 16 up to isomorphism.

P.T.O.

8. a) Prove that the group $\mathbb{Z}_m \times \mathbb{Z}_n$ is isomorphic to $\mathbb{Z}_{mn}$ if and only if m and n are relatively prime.

   b) Let X be a G-set and let $x \in G$. Prove that $|G_x| = (G : G_x)$. Also if $|G|$ is finite, show that $|G_x|$ is a divisor of $|G|$.

9. a) State and prove First Sylow theorem.

   b) Prove that no group of order 96 is simple.

Unit – II

10. Prove that any integral domain D can be embedded in a field F such that every element of F can be expressed as a quotient of two elements of D.

11. a) State and prove Third isomorphism theorem.

    b) Define free Abelian group. Prove that $\mathbb{Z}_n$ is not free Abelian.

    c) Let $G \neq \{0\}$ be a free abelian group with a finite basis. Prove that every basis of G is finite and all bases of G have the same number of elements.

12. State and prove Schreier theorem.

Unit – III

13. a) State and prove division algorithm for F[x].

    b) State Factor theorem and factorize $x^4 + 3x^3 + 2x + 4 \in \mathbb{Z}_5[x]$.

14. a) Let $f(x) \in F[x]$ and let f(x) be of degree 2 or 3. Prove that f(x) is reducible over F if and only if it has a zero in F.

    b) State and prove Eisenstein criterion for irreducibility.

    c) Prove that $25x^5 - 9x^4 - 3x^2 - 12$ is irreducible over $\mathbb{Q}$.

15. a) Let R = $\{a + b\sqrt{2} / a, b \in \mathbb{Z}\}$ and let R′ consists of all 2 × 2 matrices of the form $\begin{bmatrix} a & 2b \\ b & a \end{bmatrix}$ for $a, b \in \mathbb{Z}$. Show that R is a subring of $\mathbb{R}$ and R′ is a subring of $M_2(\mathbb{Z})$. Also show that $\phi : R \to R'$, where $\phi(a + b\sqrt{2}) = \begin{bmatrix} a & 2b \\ b & a \end{bmatrix}$ is an isomorphism.

    b) Let R be a commutative ring with unity. Prove that M is a maximal ideal of R if and only if R/M is a field.

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K21P 4210

Reg. No. : ....

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I Semester M.Sc. Degree (CBSS – Reg./Supple./Imp.) Examination, October 2021
(2018 Admission Onwards)
MATHEMATICS
MAT1C02 : Linear Algebra

Time : 3 Hours Max. Marks : 80

PART – A

Answer **four** questions from this part. **Each** question carries **4** marks.

1. Let T be a linear operator on R^3 defined by
   $T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3)$
   What is the matrix of T in the standard ordered basis for R^4.

2. Let V be a finite dimensional vector space over the field F and let W be a subspace of V. Then prove that dim W + dim W^0 = dim V.

3. Find a 3 $\times$ 3 matrix for which the minimal polynomial is x^2.

4. Let W be an invariant subspace for T. The characteristic polynomial for the restriction operator T_W divides the characteristic polynomial for T. Then prove that the minimal polynomial for T_W divides the minimal polynomial for T.

5. Let T be the linear operator on R^3 which is represented in the standard ordered basis by the matrix $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. Prove that T has no cyclic vector.

6. Apply the Gram-Schmidt process to the vectors $\beta_1 = (3, 0, 4)$, $\beta_2 = (-1, 0, 7)$, $\beta_3 = (2, 9, 11)$, to obtain an orthonormal basis for R^3 with standard inner product.

PART – B

Answer **4** questions from this part without omitting **any** Unit. **Each** question carries **16** marks.

Unit – I

7. a) Let T be a linear transformation from V into W. Then prove that T is non-singular if and only if T carries each linearly independent subset of V onto a linearly independent subset of W.

   b) If S is any subset of a finite dimensional vector space V, then prove that $(S^\circ)^\circ$ is a subspace spanned by S.

8. a) Let V be a finite-dimensional vector space over the field F and let $\{\alpha_1, \ldots, \alpha_n\}$ be an ordered basis for V. Let W be a vector space over the same field F and let $\beta_1, \ldots, \beta_n$ be any vectors in W. Then prove that there is precisely one linear transformation T from V into W such that $T\alpha_j = \beta_j$, $j = 1, \ldots, n$.

   b) If A is an $m \times n$ matrix with entries in the field F, then prove that row rank (A) = column rank (A).

9. a) Let V be an n-dimensional vector space over the field F and let W be an m-dimensional vector space over F. Then prove that the space L(V, W) is finite-dimensional and has dimension mn.

   b) Let $B = \{\alpha_1, \alpha_2, \alpha_3\}$ be the basis for C^3 defined by $\alpha_1 = (1, 0, -1)$, $\alpha_2 = (1, 1, 1)$, $\alpha_3 = (2, 2, 0)$. Find the dual basis of B.

Unit – II

10. a) Let T be a linear operator on a finite-dimensional space V. Let $c_1, \ldots, c_k$ be the distinct characteristic values of T and let W_i be the null space of $(T - c_iI)$. Then prove that following are equivalent.

    i) T is diagonalizable.

    ii) The characteristic polynomial for T is $f = (x - c_1)^{d_1} \ldots (x - c_k)^{d_k}$ and $\dim W_i = d_i$, $i = 1, \ldots, k$.

    iii) $\dim W_1 + \ldots + \dim W_k = \dim V$.

    b) Prove that similar matrices have the same characteristic polynomial.

11. a) State and prove Cayley – Hamilton Theorem.

b) Let T be the linear operator on R^2, the matrix of which in the standard ordered basis is $\begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}$. Prove that the only subspaces of R^2 invariant under T are R^2 and the zero subspace.

12. a) Let $\mathscr{F}$ be a commuting family of diagonalizable linear operators on the finite-dimensional vector space V. Prove that there exists an ordered basis for V such that every operator in $\mathscr{F}$ is represented in that basis by a diagonal matrix.

b) Find a projection E which projects R^2 onto the subspace spanned by (1, –1) along the subspace spanned by (1, 2).

Unit – III

13. a) State and prove primary decomposition theorem.

b) If V is the space of all polynomials of degree less than or equal to n over a field F, prove that the differentiation operator on V is nil potent.

14. a) Let F be a field and let B be an $n \times n$ matrix over F. Then prove that B is similar over the field F to one and only one matrix which is in rational form.

b) Let T be the linear operator on R^3 which is represented in the standard ordered basis by the matrix $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Prove that T has no cyclic vector. What is the T-cyclic subspace generated by the vector (1, –1, 3) ?

c) Verify that the standard inner product on F^n is an inner product.

15. a) Let V be an inner product space and let $(\beta_1, \ldots, \beta_n\}$, be any independent vectors in V. Then construct orthogonal vectors $\alpha_1, \ldots, \alpha_n$ in V such that for each k = 1, 2, . . ., n the set $\{\alpha_1, \ldots, \alpha_k\}$ is a basis for the subspace spanned by $\beta_1, \ldots, \beta_k$.

b) Let V be a real or complex vector space with an inner product. Show that the quadratic form determined by the inner product satisfies the parallelogram law $|| \alpha + \beta ||^2 + || \alpha - \beta ||^2 = 2 || \alpha ||^2 + 2|| \beta ||^2$.

K22P 1602

Reg. No. : ....

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I Semester M.Sc. Degree (CBSS – Reg./Sup./Imp.) Examination, October 2022
(2019 Admission Onwards)
MATHEMATICS
MAT1C02 : Linear Algebra

Time : 3 Hours

Max. Marks : 80

PART – A

Answer **any four** questions from this Part. **Each** question carries **4** marks.

1. Find a basis for a Vector Space $V = \{(x, y, z) \in R^3 / y = z + x\}$.
2. Let F be a Field and let T be a operator on R^3 defined by $T(x, y, z) = (x + 2y, x + y + z, 2y + 4z)$. Find the Matrix of T with respect to standard basis.
3. Show that similar matrices have same characteristic polynomial.
4. Let T be a linear operator on V. Show that range of T and null space of T are invariant under T.
5. **True** or **False**. Justify. "Every inner product space is a metric space".
6. Let W be a subspace of R^4 consisting of all vectors which are orthogonal to both $x = (1, 0, -1, 1)$ and $y = (2, 3, -1, 2)$. Find a basis for W.

PART – B

Answer **any four** questions from this Part without omitting **any** Unit. **Each** question carries **16** marks.

Unit – I

7. A) Define rank and nullity of a linear transformation. Let V and W be vector spaces over the field F and T be a linear transformation from V into W. Suppose V is finite dimensional then show that rank (T) + Nullity (T) = dimV.

   B) Let T is a function from R^2 into R^2 defined by $T(x, y) = (y, x)$. Check whether T is a linear transformation or not ?

P.T.O.

8. A) Let V and W be Vector Spaces over the field F and let T is a linear transformation from V into W. If T is invertible then show that, the inverse function T^{-1} is a linear transformation from W onto V.

B) Give an example of a linear transformation, which is not onto and non-singular. Also give an example of a linear transformation, which is singular and not onto.

9. A) Let V and W are Vector Spaces over the field F and let T is a linear transformation from V into W. The null space of T^t is the annihilator or range of T. If V and W are finite dimensional then show that

i) Rank (T^t) = Rank (T)

ii) The range of T^t is the annihilator of the null space of T.

B) Let A be an $m \times n$ matrix over the Field F. Then show that row rank of A = column rank of A.

Unit – II

10. State and prove Cayley-Hamilton Theorem.

11. A) Let V be finite dimensional vector space over a field F and let T be a linear operator on V. Then show that T is triangulable if and only if the minimal polynomial for T is the product of linear polynomials over F.

B) Let V be finite dimensional vector space over a field F and let T be a linear operator on V. Then show that T is diagonalizable if and only if the minimal polynomial for T has the form

$p = (x - C_1) \ldots\ldots\ldots (x - C_k)$, where $C_1, \ldots\ldots C_k$ are distinct elements of F.

12. A) Find the minimal polynomial for the 4 × 4 matrices $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$.

B) Find an invertible real matrix p such that P^{-1} AP are diagonal.

Where $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 0 & 4 \end{bmatrix}$.

Unit – III

13. A) Let T be a linear operator on a finite dimensional vector space V over the field F. Suppose that minimal polynomial for T decomposes over into a product of linear polynomials. Show that there is a diagonalizable operator D on V and a nilpotent operator N on V such that

    i) T = D + N

    ii) DN = ND

    D and N are uniquely determined by (i) and (ii) and each of them is a polynomial in T.

    B) If A is the companion matrix of a monic polynomial p, then show that p is the minimal and characteristic polynomial of A.

14. A) Let A be a complex 3 × 3 matrices given by $A = \begin{bmatrix} 2 & 0 & 0 \\ a & 2 & 0 \\ b & c & -1 \end{bmatrix}$. Show that A is similar to diagonal matrix if and only if a = 0.

    B) Let V be the space of all n-time differentiable functions on an interval of real line which satisfying the differential equation $\frac{d^n f}{dx^n} + a_{n-1}\frac{d^{n-1}f}{dx^{n-1}} + \ldots\ldots\ldots + a_1\frac{d^1 f}{dx^1} + a_0 f = 0$, where the coefficients are complex numbers. Let D be the differential operator on V. What is the Jordan form for the differentiation operator on V ?

15. A) Define inner product space. Give an example of inner product space.

    B) State and prove polarization identities in real and complex cases.

    C) Show that every finite dimensional inner product has an orthonormal basis.

K21P 4211

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I Semester M.Sc. Degree (C.B.S.S. – Reg./Supple./Imp.) Examination, October 2021
(2018 Admission Onwards)
MATHEMATICS
MAT1C03 : Real Analysis

Time : 3 Hours   Max. Marks : 80

PART – A

Answer **any four** questions from this Part. **Each** question carries **4** marks :

1. Let A be the set of all sequences whose elements are the digits 0 and 1. Show that A is countable.
2. If f is monotonically increasing on (a, b), show that f(x –) exists and $f(x-) \leq f(x)$ for every $x \in (a, b)$.
3. Let $f(x) = x^{10} \sin\frac{1}{x}$ if $x \neq 0$ and f(0) = 0. Is f differentiable at all points ? If so, find $f'(x)$ for all x.
4. If f is continuous on [a, b], show that $f \in R(\alpha)$ on [a, b].
5. State and prove the integration by parts theorem.
6. Is the curve $f(t) = e^{2\pi it}$, $t \in [0, 2]$ rectifiable ? Justify. If rectifiable, find its arc length.

PART – B

Answer **any four** questions from this Part without omitting **any** Unit. **Each** question carries **16** marks :

Unit – I

7. a) Suppose X is a metric space and let $K \subset Y \subset X$. Show that K is compact relative to X if and only if K is compact relative to Y.

   b) Construct the Cantor set and show that it is perfect.

   c) If f is a continuous mapping of a metric space X into a metric space Y and if E is a connected subset of X, show that f(E) is connected.

P.T.O.

8. a) Show that every K-cell is compact.

   b) Show that a mapping f of a metric space X into a metric space Y is continuous if and only if $f^{-1}(V)$ is open in X for any open set V in Y.

9. a) Prove that a subset E of the real line $\mathbb{R}$ is connected if and only if it has the following property : if $x \in E$, $y \in E$ and $x < z < y$, then $z \in E$.

   b) Let f be a continuous mapping of a compact metric space X into a metric space Y. Show that f is uniformly continuous on X.

Unit – II

10. a) State and prove L'Hospital's Rule.

    b) Assume α increases monotonically and $\alpha' \in R$ on [a, b]. Let f be a bounded real function on [a, b]. Show that, $f \in R(\alpha)$ if and only if $f\alpha' \in R$ and in that case,

    $\int_a^b f\,d\alpha = \int_a^b f(x)\alpha'(x)dx$.

11. a) Suppose $f \in R(\alpha)$ on [a, b] and let $m \le f \le M$. A function ϕ is continuous on [m, M] and $h(x) = \phi(f(x))$ on [a, b]. Show that $h \in R(\alpha)$ on [a, b].

    b) Suppose f is bounded on [a, b]. If f has only finitely many points of discontinuity on [a, b] and if α is continuous at any point at which f is continuous, show that $f \in R(\alpha)$.

    c) Suppose $f : [a, b] \to \mathbb{R}^k$ is continuous and f is differentiable in (a, b). Show that there exists $x \in (a, b)$ such that $|f(b) - f(a)| \le (b - a)\,|f'(x)|$.

12. a) State and prove change of variable rule in Riemann-Stieltjes integration.

    b) State and prove the generalized mean value theorem and deduce the mean value theorem.

    c) Let f and α be functions on $\left[0, \frac{\pi}{2}\right]$ defined as $f(x) = \cos x$, $\alpha(x) = \sin x$.

    Is $f \in R(\alpha)$? Justify. If $f \in R(\alpha)$ evaluate $\int_0^{\pi/2} f d\alpha$.

Unit – III

13. a) Let $f \in R$ on [a, b]. For $a \leq x \leq b$, let $F(x) = \int_a^x f(t)\,dt$. Show that F is continuous on [a, b]. Furthermore, if f is continuous at a point x_0 of [a, b], then show that F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

    b) Let f be of bounded variation on [a, b]. Let $V(x) = V_f(a, x)$ if $a < x \leq b$ and $V(a) = 0$. Show that every point of continuity of f is also a point of continuity of V. Prove the converse also.

    c) Let $f : [a, b] \to \mathbb{R}$ satisfies $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in [a, b]$ and $K > 0$. Is f of bounded variation ? Justify.

14. a) If $f : [a, b] \to \mathbb{R}^k$ and if $f \in R(\alpha)$ for some monotonically increasing α on [a, b], show that $|f| \in R(\alpha)$ and $\left|\int_a^b f\,d\alpha\right| \leq \int_a^b |f|\,d\alpha$.

    b) State and prove additive property of arc length.

    c) If f is monotone increasing on [a, b], evaluate the total variation of f on [a, b].

15. a) State and prove fundamental theorem of calculus.

    b) Let $f : [a, b] \to \mathbb{R}^n$ be a rectifiable path. If $x \in (a, b]$, let $s(x) = \wedge_f(a, x)$ and let $s(a) = 0$. Show that the following holds :

       i) The function s is increasing and continuous on [a, b].

       ii) If there is no subinterval of [a, b] on which f is constant, then s is strictly increasing on [a, b].

    c) Is the function $f(x) = x \sin\frac{\pi}{x}$ if $x \neq 0$ and $f(0) = 0$ is of bounded variation on [0, 1] ? Justify.

K21P 4213

Reg. No. : ....

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I Semester M.Sc. Degree (CBSS – Reg./Supple./Imp.) Examination, October 2021
(2018 Admission Onwards)
MATHEMATICS
MAT1C05 : Differential Equations

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any four** questions from this Part. **Each** question carries **4** marks **each**.

1. Find a power series solution of the differential equation $y' = 2xy$.

2. Locate and classify the singular points of

   i) $x^2\left(x^2-1\right)^2 y'' - x(1-x)y' + 2y = 0$

   ii) $x^4y'' + (\sin x)y = 0$.

3. State the generating function for the Legendre polynomial $P_n(x)$. Use it to prove that $P_{2n}(0) = (-1)^n \dfrac{1.3.\cdots(2n-1)}{2^n n!}$.

4. Prove that $J_{\frac{1}{2}}(x) = \sqrt{\dfrac{2}{\pi x}}\sin x$.

5. Obtain the normal form of Bessel equation $x^2y'' + xy' + (x^2 - p^2)y = 0$.

6. Find the first three approximate solutions of the initial value problem $y' = y^2$, $y(0) = 1$ using Picard's method.

P.T.O.

PART – B

Answer **any four** questions from this Part without omitting any Unit. **Each** question carries **16** marks **each**.

UNIT – 1

7. a) Find the general solution of $(1+x^2)y'' + 2xy' - 2y = 0$ in terms of power series in x. **7**

   b) Solve the differential equation $2x^2y'' + x(2x+1)y' - y = 0$. **9**

8. a) Find the indicial equation and its roots of the differential equation $4x^2y'' + (2x^4 - 5x)y' + (3x^2 + 2)y = 0$. **6**

   b) Find two independent Frobenius solutions of the equation $x^2y'' - x^2y' + (x^2 - 2)y = 0$. **10**

9. Find the general solution of the Gauss Hypergeometric differential equation. **16**

UNIT – 2

10. a) If $P_m(x)$ and $P_n(x)$ respectively are m^{th} and n^{th} Legendre polynomials, then prove that

$$\int_{-1}^{1} P_m(x)\,P_n(x)dx = \begin{cases} 0 & \text{if} \quad m \neq n \\ \dfrac{2}{2n+1} & \text{if} \quad m = n \end{cases}$$

10

   b) Obtain the recursion formula $(n + 1)P_{n+1}(x) = (2n + 1)xP_n(x) - nP_{n-1}(x)$. **6**

11. a) Solve the Bessel equation $x^2y'' + xy' + (x^2 - p^2)y = 0$ to get the Bessel function of first kind of order p. **9**

   b) Prove that :

   i) $\dfrac{d}{dx}J_0(x) = -J_1(x)$

   ii) $\dfrac{d}{dx}xJ_1(x) = xJ_0(x)$. **7**

12. a) Find the general solution of the system of homogeneous equations

$$\frac{dx}{dt} = x + y$$

$$\frac{dy}{dt} = 4x - 2y$$

9

b) If W(t) is the Wronskian of two solutions $\begin{cases} x = x_1(t) \\ y = y_1(t) \end{cases}$ and $\begin{cases} x = x_2(t) \\ y = y_2(t) \end{cases}$ of the homogeneous system of equations

$$\frac{dx}{dt} = a_1(t)x + b_1(t)y$$

$$\frac{dy}{dt} = a_2(t)x + b_2(t)y$$

then prove that W(t) of solutions is either identically zero or nowhere zero on [a, b]. **7**

UNIT – 3

13. a) If $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of $y'' + P(x)y' + Q(x)y = 0$, then prove that the zeros of these functions are distinct and occur alternatively in the sense that $y_1(x)$ vanishes exactly once between any two successive zeros of $y_2(x)$ and conversely. **8**

b) Let u(x) be any non-trivial solution of $u'' + q(x)u = 0$, where q(x) > 0 for all x > 0. If $\int_0^\infty q(x)dx = \infty$, then prove that u(x) has infinitely many zeros on the positive x-axis. **8**

14. Let f(x, y) be a continuous function that satisfies Lipschitz condition $|f(x, y_1) - f(x, y_2)| \le K|y_1 - y_2|$ on a strip defined by $a \le x \le b$ and $-\infty \le y \le \infty$. If (x_0, y_0) is any point of the strip, prove that the Initial Value Problem $y' = f(x, y)$, $y(x_0) = y_0$ has one and only one solution y = y(x) on the interval $a \le x \le b$. **16**

15. a) Show that f(x, y) = xy satisfy Lipschitz condition on any rectangle $a \le x \le b$, $c \le y \le d$. Also prove that f(x, y) does not satisfy the Lipschitz condition in the entire plane. **8**

b) Solve the Initial Value Problem using Picard's method of successive method of approximation.

$$\frac{dy}{dx} = z \qquad y(0) = 1$$

$$\frac{dz}{dx} = -y \qquad z(0) = 0.$$ **8**