A complex number z is an ordered pair (x, y) of real numbers. i.e. $z=(x,y)$ $x\in\mathbb{R},\ y\in\mathbb{R}$. Complex number system, denoted by $\mathbb{C}$ is the set of all ordered pairs of real numbers (i.e. $\mathbb{R}\times\mathbb{R}$) with the two operations of addition and multiplication ($\cdot$ or $\times$) which satisfy :

(i) $(x_1, y_1)+(x_2, y_2)=(x_1+x_2+y_1+y_2)$

(ii) $(x_1, y_1).(x_2, y_2)=(x_1 x_2 - y_1 y_2,\ x_1 y_2 + x_2 y_1)$

$\forall (x_1, y_1), (x_2, y_2)\in\mathbb{C}$

The word ordered pair means (x_1, y_1) and (y_1, x_1) are distinct unless $x_1 = y_1$.

Let $z=(x,y)$; $x\in\mathbb{R},\ y\in\mathbb{R}$. '$x$' is called Real part of a complex number z and it is denoted by $x=\operatorname{Re} z$, (Real part of z) and 'y' is called Imaginary part of z and it is denoted by $y=\operatorname{Im} z$.

Two complex numbers $z_1=(x_1, y_1)$ and $z_2=(x_2, y_2)$ are said to be equal iff $x_1=x_2$ and $y_1=y_2$ i.e. real part and imaginary part both are equal.

About Symbol 'i':

The complex number $(0, 1)$ is denoted by 'i' and is called the imaginary number.

$i^2 = i.i = (0,1).(0,1)$

$= (0-1,\ 0+0)$ by property (ii) abov

$= (-1, 0)$

$\Rightarrow$ $\boxed{i^2=-1}$

Similarly,

$i^3 = i^2.i = (-1,0).(0,1) = (0-0,\ -1+0) = (0,-1)$

$\Rightarrow$ $\boxed{i^3=-i}$

$i^4 = i^3.i = (0,-1).(0,1) = (0+1,\ 0+0) = (1,0) \therefore \sqrt{3}$

$\Rightarrow$ $\boxed{i^4=1}$

Using this symbol i, we can write a complex number (x, y) as $x+iy$ (Since $x+iy=(x,0)+(0,1)(y,0)=(x,0)+(0,y)=(x,y)$ The complex number $z=(x,y)$ can be written as $z=x+iy$

Note: (The set of all complex numbers) $\mathbb{C}$ forms a field.

Propeties of complex numbers

Let $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ and $z_3 = (x_3, y_3) \in \mathbb{C}$.

1) **Closure Law :** $z_1 + z_2 \in \mathbb{C}$ and $z_1 . z_2 \in \mathbb{C}$

2) **Commutative Law of addition** : $z_1 + z_2 = z_2 + z_1$

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2,\ y_1 + y_2) = (x_2 + x_1,\ y_2 + y_1)$$
$$= (x_2, y_2) + (x_1, y_1) = z_2 + z_1$$

3) **Associative Law of addition** : $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

$$z_1 + (z_2 + z_3) = (x_1, y_1) + [(x_2, y_2) + (x_3, y_3)]$$
$$= (x_1, y_1) + (x_2 + x_3,\ y_2 + y_3) \qquad = (x_1 + x_2 + x_3,\ y_1 + y_2 + y_3)$$
$$= (x_1 + x_2, y_1 + y_2) + (x_3 + y_3) = [(x_1, y_1) + (x_2, y_2)] + (x_3, y_3)$$
$$= (z_1 + z_2) + z_3$$

4) **Existence of additive Identity** : The Complex Number $0 = (0, 0)$ i.e. $z = 0 + 0i$ is called the identity with respect to addition.

5) **Existence of additive Inverse** :
For each complex number $z_1 \in \mathbb{C}$, $\exists$ a unique complex number $z \in \mathbb{C}$ s.t. $z_1 + z = z + z_1 = 0$ i.e. $z = -z_1$. The complex number z is called the additive inverse of z_1 and it is denoted by $z = -z_1$.

6) **Commutative law of Multiplication** : $z_1 . z_2 = z_2 . z_1$

$$z_1 . z_2 = (x_1, y_1).(x_2, y_2) = (x_1 \cdot x_2 - y_1 \cdot y_2, x_1 \cdot y_2 + x_2 \cdot y_1) \quad \ldots\ldots (1)$$

and

$$z_2 . z_1 = (x_2, y_2).(x_1, y_1) = (x_2 . x_1 - y_2\, y_1, x_2 . y_1 + x_1 . y_2)$$
$$= (x_1 . x_2 - y_2 . y_1,\ x_1 . y_2 + x_2 . y_1) = z_1 . z_2 \qquad \text{from (1)}$$

7) **Associative Law of Multiplication** : $z_1 .(z_2 . z_3) = (z_1 . z_2). z_3$

$$z_1 .(z_2 . z_3) = (x_1, y_1)[(x_2, y_2).(x_3 . y_3)]$$
$$= (x_1, y_1)[x_2 . x_3 - y_2 . y_3, x_2 . y_3 + x_3 . y_2]$$
$$= [x_1(x_2 . x_3 - y_2\, y_3) - y_1(x_2 . x_3 - y_2 . y_3),\ x_1(x_2 . x_3 + x_3 . y_2) + y_1(x_2 . x_3 - y_2 . y_3)]$$

$$= (x_1 . x_2 . x_3 - x_1 . y_2 . y_3 - x_1 . x_3 . y_1 + y_1 . y_2 . y_3,\ \ x_1 . x_2 . x_3 + x_1 . x_3 . y_2$$
$$+ x_2 . x_3 . y_1 + y_1 . y_2 . y_3) \qquad (*)$$

$$(z_1 . z_2) . z_3 = [(x_1, y_1) . (x_2, y_2)](x_3, y_3) = (x_1 x_2 - y_1 y_2,\ x_1 y_2 + x_2 y_1)(x_3, y_3)$$

$$= x_1 x_2 x_3 - x_3 y_1 y_2 - x_1 x_2 y_3 + y_1 y_2 y_3,\ x_1 x_3 y_2 + x_2 y_1 y_3 + x_1 x_2 x_3 - y_1 y_2 y_3$$

$$= z_1 (z_2 . z_3) \qquad \text{from } (*)$$

8) **<u>Existence of Multiplicative Identity</u> :** $z_1 . 1 = 1 . z_1 = z_1$

The complex number $1 = (1, 0)$ (i.e. $z = 1 + 0i$) is called the identity with respect to multiplication.

9) **<u>Existence of Multiplicative Inverse</u> :** For each complex number $z_1 \neq 0$, there exists a unique complex number z in $\mathbb{C}$ s.t. $z_1 . z = z . z_1 = 1$ i.e. $z = \dfrac{1}{z_1}$ is called the multiplicative inverse of complex number z_1 and it is denoted by $z = \dfrac{1}{z_1}$ or z^{-1}.

Let $z = (x, y)$ and $z_1 = (x_1, y_1)$

$\therefore\ z_1 . z_2 = 1$

$\therefore\ (x, y)(x_1, y_1) = (1, 0) \quad \Rightarrow (x x_1 - y y_1,\ x y_1 + x_1 y) = (1, 0)$

$\Rightarrow\ x . x_1 - y . y_1 = 1$ ……..(i) and $x . y_1 + x_1 . y = 0$ ……...(ii)

Equation (ii) $\times x_1$ - Equation (i) $\times y_1$, we get

$$\begin{array}{rcrcl} & x x_1 y_1 & + & x_1^2 y & = 0 \\ - & x x_1 y_1 & - & y y_1^2 & = y_1 \\ \hline & - & + & & - \\ \end{array}$$

$$y\left(x_1^2 + y_1^2\right) = -y_1$$

$$\therefore \boxed{y = \frac{-y_1}{x_1^2 + y_1^2}} \qquad \text{(iii)}$$

Substitute equation (iii) in equation (ii) i.e. $x . y_1 + y . x_1 = 0$

$$\therefore\ x . y_1 = -y . x_1 = -\left(\frac{-y_1}{x_1^2 + y_1^2}\right) x_1 \Rightarrow x = \frac{x_1\, y_1}{x_1^2 + y_1^2} \times \frac{1}{y_1}$$

$$\boxed{x = \frac{x_1}{x_1^2 + y_1^2}}$$

$$\therefore\quad z = \left(\frac{x_1}{x_1^2 + y_1^2}, \frac{-y_1}{x_1^2 + y_2^2}\right)$$

z is the multiplicative inverse of complex number $z_1 = (x_1, y_1)$.

10) Distributive Law :

$$z_1(z_2+z_3) = z_1 . z_2 + z_1 . z_3$$

Subtraction: The difference of two complex Numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ is defined as :

$$z_1 - z_2 = (x_1, y_1) - (x_2, y_2) = (x_1 - x_2,\ y_1 - y_2)$$

Division: It is defined by the equality $\frac{z_1}{z_2} = z_1 . z_2^{-1} \quad z_2 \neq 0$

$$= (x_1, y_1)\left(\frac{x_2}{x_2^2 + y_2^2}, \frac{-y_2}{x_2^2 + y_2^2}\right) = \left(\frac{x_1 . x_2 + y_1 . y_2}{x_2^2 + y_2^2}, \frac{-x_1 y_2 + x_2 y_1}{x_2^2 + y_2^2}\right)$$

Geometrical Representation of a Complex Number :

Consider a complex number $z = x + iy$.

$\because$ Complex number is defined as an ordered paired of real numbers.

i.e. $z = (x,\ y)$

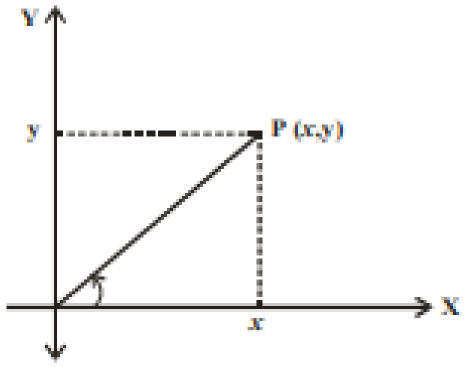

Fig. 1.1

This form of a complex number z suggest that z can be represented by point (say) *P* whose Cartesian co-ordinates are *x* and *y* referred (relating) to rectangular axis *X* and *Y*, usually called the Real and Imaginary axis respectively.

To each complex number there corresponds points in the plane and conversely, one and only one each point in the plane there exist one and only one complex number.

A plane whose points are represented by the complex numbers is called Complex Plane or Gaussian Plane or Argand Plane. Gauss was first who formulated that complex numbers are represented by points in a plane in 1799 then in 1806 it was done by Argand.

Vector Representation of a Complex Numbers :

If P is the point in the Complex Plane corresponding to complex number z can be considered as vector $\overline{OP}$ whose initial point is the origin 'O' and terminal point is $P = z = (x, y)$ as shown in the figure 1.2.

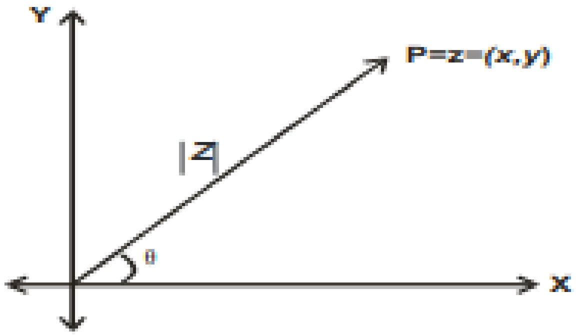

Fig 1.2

Conjugate:

If $z = x + iy \in \mathbb{C}$ then the complex number $x - iy$ is called the conjugate of a complex number z or complex conjugate and it is denoted by $\bar{z}$.

e.g. $z = 4 + 3i \quad \Rightarrow \bar{z} = 4 - 3i$

$w = 4 + 5e^{3i} \quad \Rightarrow \bar{w} = 4 + 5e^{-3i}$

Geometrically:

The complex conjugate of a complex number $z = (x, y)$ is the image or reflection of z in the real axis.

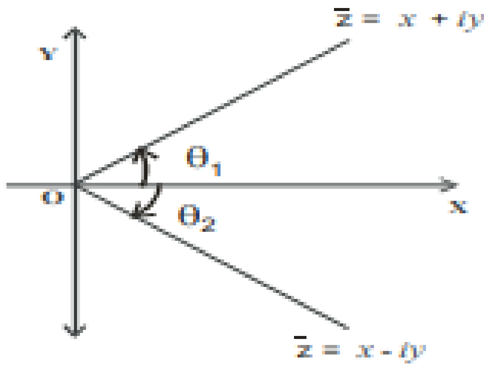

Fig 1.3

Let $z = x + iy$

$\because$ $x = \operatorname{Re} z$ and $z = \operatorname{Im} z$. $\therefore$ $x = \operatorname{Re} z = \dfrac{z + \bar{z}}{2}$ *and* $y = \operatorname{Im} z = \dfrac{z - \bar{z}}{2}$

Definition: The modulus or absolute value of a complex number $z = x + iy$ is defined by $|z| = \sqrt{x^2 + y^2}$.

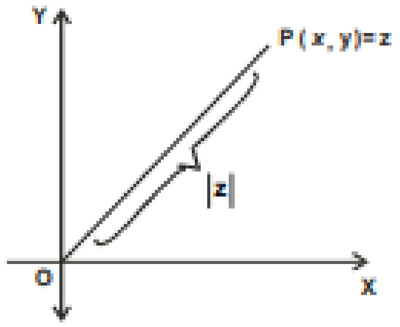

Fig 1.4

The distance between Two Complex Numbers :

Let $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ in complex plane is given by

$d(z_1, z_2) = |z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

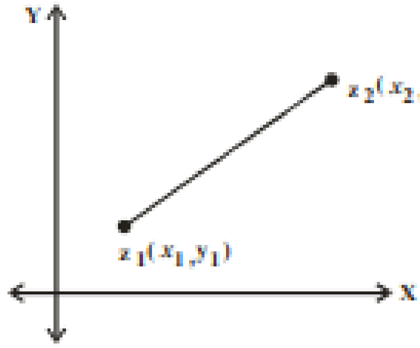

Fig 1.5

Polar form of a Complex Numbers :

If P is a point in the Complex Plane corresponding to complex number $z = x + iy = (x, y)$ and let (r, θ) be the polar co-ordinates of point (x, y) from figure 1.3, $x = r\cos\theta$ and $y = r\sin\theta$, where $r = \sqrt{x^2 + y^2}$ is called the modulus or absolute value of z (denoted by $|z|$ and $\boxed{\theta = \tan^{-1}\left(\frac{y}{x}\right)}$ is called the argument or amplitude of z (denoted by $\theta = \arg z$). Here θ is the angle between the two lines OP and the real axis (x- axis) .

Exponential form of complex number: A complex number can be written in the form of $z = re^{i\theta}$, where $r = \sqrt{x^2 + y^2}$ and

$$\boxed{\theta = \tan^{-1}\left(\frac{y}{x}\right)}$$

.This is known as the exponential form.

(Note: $e^{i\theta} = \cos\theta + i\sin\theta$,known as Euler's Identity)

Note : 1) $\left|e^{i\theta}\right| = 1$, 2) $\left|e^{\frac{i500}{4}\pi}\right| = 1$

Solved Examples :

1. Let $z_1 = 1 + i, z_2 = 1 - 2i, z_3 = 1 + \sqrt{3}i$. Find i) $z_1.z_2$ ii) z_1/z_2 iii) $\bar{z}_2$ iv) $|z_1|$ v) $arg(z_1)$ vi) Express z_1 in polar and exponential form.

Solution:

i) $z_1z_2 = (1+i)(1-2i) = 1 - 2i - i - 2i^2 = 1 - 2i - i + 2 = 3 - 3i$

ii) $z_1 / z_2 = \dfrac{1+i}{1-2i} = \dfrac{1+i}{1-2i} \times \dfrac{1+2i}{1+2i} = \dfrac{1+2i+i-2}{1-4i^2} = \dfrac{-1+3i}{5}$

iii) $\bar{z}_2 = 1+2i$

iv) $x_1 = 1, y_1 = 1$

$$\therefore |z_1| = \sqrt{x^2 + y^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$$

v) $x_1 = 1, y_1 = 1$

$$\therefore \theta = \tan^{-1}\left(\frac{y_1}{x_1}\right) = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

vi) $z_1 = r_1(\cos\theta_1 + \sin\theta_1)$

$$\therefore z_1 = \sqrt{2}\left(\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\right)$$

2. Find the principal value of $\arg 'i'$

$\theta = Arg\ z \qquad -\pi < \theta \le \pi$

$$Arg\ i = \theta = \tan^{-1}\left(\frac{y}{x}\right)\{\because z = i \Rightarrow z = (o.x + iy)\}$$

$$= \tan^{-1}\left(\frac{1}{0}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$$

3. Find the principal value of $\arg(1+i)$

$\because z = 1 + i = (x + iy)$

$\therefore x = 1, \ y = 1$

$$Arg\ z = \theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1}{1}\right) = \boxed{\frac{\pi}{4}}$$

Properties of polar form and exponential form

1) Let $z_1 = r_1 e^{i\theta_1} = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2 e^{i\theta_2} = r_2(\cos\theta_1 + i\sin\theta_2)$

then $z_1 . z_2 = r_1 . r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = r_1 . r_2 \, e^{i(\theta_1+\theta_2)}$

$\because \; e^{i\theta} = e^{i(\theta+2n\pi)}, \quad e^{2in\pi} = 1, \quad n \in \mathbb{Z}$

$|z_1 . z_2| = \left| r_1 . r_2 . e^{i(\theta_1+\theta_2)} \right| = r_1 . r_2 \qquad \left(\because \left| e^{i(\theta_1+\theta_2)} \right| = 1 \right)$

and $\boxed{\arg(z_1 . z_2) = \arg z_1 + \arg z_2 \pmod{2\pi}}$ in the sense that they are same but for an integral multiple of 2π.

Note : $\boxed{\arg z_1 . z_2 = \arg z_1 + \arg z_2 + 2k\pi \qquad \text{where } k = 0, 1 \text{ or } -1}$

2. Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ and $z_2 \neq 0$

$\therefore \; \dfrac{z_1}{z_2} = \dfrac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \dfrac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$

$\therefore \; \arg\left(\dfrac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 \pmod{2\pi}$

Let $z_1 = -1$ and $z_2 = -i$, $\quad (-\pi \le \theta \le \pi)$

$\because \; z_1 = -1 = x + iy \Rightarrow x = -1$ and $y = 0$

$\therefore \quad \arg z_1 = \arg(-1) = \tan^{-1}\left(0/_{-1}\right) = \tan^{-1}(0) = \tan^{-1}(\tan\pi) = \pi$

$\because \quad z_2 = -i = x + iy \quad \Rightarrow \; x = 0$ and $y = -1$

$\therefore \quad \arg z_2 = \arg(-i) = \tan^{-1}\left(y/_x\right) = \tan^{-1}\left(-1/_0\right) = \tan^{-1}(\infty) = -\dfrac{\pi}{2}$

$\arg(z_1 . z_2) = \arg(-1 . -i) = \arg(i) = \tan^{-1}\left(1/_0\right) = \tan^{-1}(\infty) = \dfrac{\pi}{2}$

$\therefore \quad \arg(z_1 . z_2) = \arg z_1 + \arg z_2 + 2k\pi \qquad$ where $k = 0$

Let $z_1 = -1$ and $z_2 = i$

$\because \quad z_1 = -1 \Rightarrow z_1 = x + iy \Rightarrow x = -1, \; y = 0$

$\therefore \quad \arg z_1 = \arg(-1) = \tan^{-1}\left(y/_x\right) = \tan^{-1}\left(0/_{-1}\right) = \tan^{-1}(0) = \pi$

$\because \quad z_2 = i = x + iy \Rightarrow x = 0, \; y = 1$

$\therefore \quad \arg z_2 = \arg(i) = \tan^{-1}\left(y/_x\right) = \tan^{-1}\left(1/_0\right) = \tan^{-1}(\infty) = \dfrac{\pi}{2}$

$\therefore \; \arg(z_1 . z_2) = \arg(-1 . i) = \arg(-i) = \tan^{-1}\left(y/_x\right) = \tan^{-1}\left(-1/_0\right)$

$= -\tan(\infty) = -\dfrac{\pi}{2}$

$\therefore \quad \arg(z_1 . z_2) = \arg z_1 + \arg z_2 + 2k\pi \quad$ where $k = -1$

In this case, we get correct answer by adding -2π to bring within the interval $(-\pi, \pi)$.

When principal argument are added together in multiplication problem, the resulting argument need not be the principle value.

<u>***De-Moivre's Theorem***</u> :

Theorem: *If* n *is any integer or fraction then* $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$

Example : Find all the fourth roots of $z = 1 + i$ and locate these roots in $\mathbb{C}$ plane.

Solution : Let $w^4 = z = 1 + i$

$\therefore\ x = 1,\ y = 1$

$\therefore\ r = \sqrt{x^2 + y^2} = \sqrt{1+1} \ \therefore \boxed{r = \sqrt{2}}$

$\theta = \tan^{-1}\left(y/x\right) = \tan^{-1}\left(1/1\right) = \tan^{-1}(1) \ \therefore \boxed{\theta = \pi/4}$

$$\therefore\ w^4 = \sqrt{2}\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right] \qquad \text{(polar form)}$$

$$= \sqrt{2}\left[\cos\left(\pi/4 + 2k\pi\right) + i\sin\left(\pi/4 + 2k\pi\right)\right]$$

$$w = 2^{1/8}\left[\cos\left(\frac{\pi + 8k\pi}{4}\right) + i\sin\left(\frac{\pi + 2k\pi}{4}\right)\right]^{1/4}$$

$$w = 2^{1/8}\left[\cos\left(\frac{\pi + 8k\pi}{16}\right) + i\sin\left(\frac{\pi + 2k\pi}{16}\right)\right] \qquad \text{where}$$

$\therefore$ Fourth roots of equations are

For $k = 0,\ w_0 = z^{1/8}\left[\cos\left(\frac{\pi}{16}\right) + i\sin\left(\frac{\pi}{16}\right)\right]$

$k = 1,\ w_1 = z^{1/8}\left[\cos\left(\frac{9\pi}{16}\right) + i\sin\left(\frac{9\pi}{16}\right)\right]$

$k = 2,\ w_2 = z^{1/8}\left[\cos\left(\frac{17\pi}{16}\right) + i\sin\left(\frac{17\pi}{16}\right)\right]$

$k = 3,\ w_3 = z^{1/8}\left[\cos\left(\frac{25\pi}{16}\right) + i\sin\left(\frac{25\pi}{16}\right)\right]$

Which are the required four fourth root of $z = 1 + i$