THERMODYNAMICS AND STATISTICAL PHYSICS

NET/JRF (JUNE-2011)

Q1. Consider the transition of liquid water to steam as water boils at a temperature of 100^0C under a pressure of 1 atmosphere. Which one of the following quantities does not change discontinuously at the transition?

(a) The Gibbs free energy (b) The internal energy

(c) The entropy (d) The specific volume

Ans. : (a)

Solution: In first order transition Gibbs free energy is continuous.

Q2. A particle is confined to the region $x \geq 0$ by a potential which increases linearly as $u(x) = u_0 x$. The mean position of the particle at temperature T is

(a) $\frac{k_B T}{u_0}$ (b) $(k_B T)^2 / u_0$ (c) $\sqrt{\frac{k_B T}{u_0}}$ (d) $u_0 k_B T$

Ans. : (a)

Solution: Partition function $Z = \frac{1}{h}\int e^{-\frac{p^2}{2mk_BT}} dp \int e^{-\frac{u_0 x}{k_B T}} dx$ and $\langle x \rangle = \int xp(x) dx dp_x$

$$\Rightarrow \langle x \rangle = \frac{\iint xe^{-\frac{p^2}{2mk_BT}} dp \cdot e^{-\frac{u_0 x}{k_BT}} dx}{\iint e^{-\frac{p^2}{2mk_BT}} dp \cdot e^{-\frac{u_0 x}{k_BT}} dx} = \frac{\int_0^\infty xe^{-\frac{\mu_0 x}{k_BT}} dx}{\int_0^\infty e^{-\frac{\mu_0 x}{k_BT}} dx} = \frac{\left(\frac{k_BT}{u_0}\right)^2 \int_0^\infty te^{-t} dt}{\left(\frac{k_BT}{u_0}\right) \int_0^\infty e^{-t} dt} = \frac{k_BT}{u_0}$$

Q3. A cavity contains blackbody radiation in equilibrium at temperature T. The specific heat per unit volume of the photon gas in the cavity is of the form $C_V = \gamma T^3$, where γ is a constant. The cavity is expanded to twice its original volume and then allowed to equilibrate at the same temperature T. The new internal energy per unit volume is

(a) $4\gamma T^4$ (b) $2\gamma T^4$ (c) γT^4 (d) $\frac{\gamma T^4}{4}$

Ans. : (d)

Solution: $du = C_v dT = \int \gamma T^3 dT \Rightarrow u = \frac{\gamma T^4}{4}$

Q4. Consider a system of N non-interacting spins, each of which has classical magnetic moment of magnitude μ. The Hamiltonian of this system in an external magnetic field $\vec{H}$ is $\sum_{i=1}^{N} \vec{\mu}_i . \vec{H}$, where $\vec{\mu}_i$ is the magnetic moment of the i^{th} spin. The magnetization per spin at temperature T is

(a) $\frac{\mu^2 H}{k_B T}$ (b) $\mu\left[\coth\left(\frac{\mu H}{k_B T}\right) - \frac{k_B T}{\mu H}\right]$

(c) $\mu \sinh\left(\frac{\mu H}{k_B T}\right)$ (d) $\mu \tanh\left(\frac{\mu H}{k_B T}\right)$

Ans. : (b)

Solution: For classical limit $M = \dfrac{\int_0^{2\pi}\int_0^{\pi} \mu \cos\theta \exp\frac{\mu H \cos\theta}{kT} \sin\theta \; d\theta \; d\phi}{\iint \exp\frac{\mu H \cos\theta}{k_B T} \sin\theta \; d\theta d\phi}$

$$M = \mu\left[\coth\left(\frac{\mu H}{k_B T}\right) - \frac{k_B T}{\mu H}\right]$$

Q5. Consider an ideal Bose gas in three dimensions with the energy-momentum relation $\varepsilon \propto p^s$ with $s > 0$. The range of s for which this system may undergo a Bose-Einstein condensation at a non-zero temperature is

(a) $1 < s < 3$ (b) $0 < s < 2$ (c) $0 < s < 3$ (d) $0 < s < \infty$

Ans. : (a)

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Q6. The internal energy E of a system is given by $E = \frac{bS^3}{VN}$, where b is a constant and other symbols have their usual meaning. The temperature of this system is equal to

(a) $\frac{bS^2}{VN}$ (b) $\frac{3bS^2}{VN}$ (c) $\frac{bS^3}{V^2 N}$ (d) $\left(\frac{S}{N}\right)^2$

Ans. : (b)

Solution: $TdS = dE + PdV \Rightarrow dE = TdS - PdV \Rightarrow \left(\frac{\partial E}{\partial S}\right)_V = T \Rightarrow T = \frac{3bS^2}{VN}$

Q7. Consider a Maxwellian distribution of the velocity of the molecules of an ideal gas. Let V_{mp} and V_{rms} denote the most probable velocity and the root mean square velocity, respectively. The magnitude of the ratio V_{mp}/V_{rms} is

(a) 1 (b) $2/3$ (c) $\sqrt{2/3}$ (d) $3/2$

Ans. : (c)

Solution: For Maxwellian distribution $V_{mp} = \sqrt{\frac{2kT}{m}}$, $V_{rms} = \sqrt{\frac{3kT}{m}} \Rightarrow \frac{V_{mb}}{V_{rms}} = \sqrt{\frac{2}{3}}$

Q8. If the number density of a free electron gas in three dimensions is increased eight times, its Fermi temperature will

(a) increase by a factor of 4 (b) decrease by a factor of 4

(c) increase by a factor of 8 (d) decrease by a factor of 8

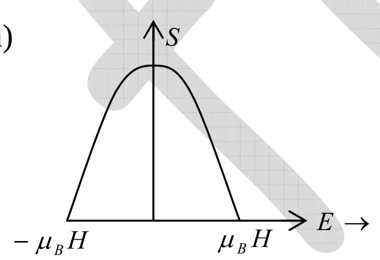
Ans. : (a)

Solution: Fermi energy $E_F = \left(\frac{3N}{4\pi Vg}\right)^{\frac{2}{3}} \frac{\hbar^2}{2m}$, where $\frac{N}{V}$ is number density and g is degeneracy

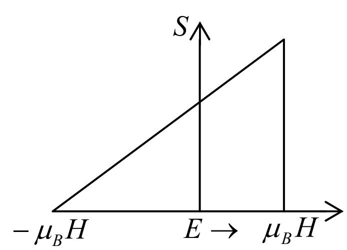
$$E_F \propto T_F K \Rightarrow T_F \propto \left(\frac{n}{V}\right)^{\frac{2}{3}} \Rightarrow T_F \propto (n)^{\frac{2}{3}} \Rightarrow \frac{T_{F_1}}{T_{F_2}} = \left(\frac{n_1}{n_2}\right)^{\frac{2}{3}} = 4 \text{ since } \frac{n_1}{n_2} = 8.$$

Q9. A system of N non-interacting spin - $\frac{1}{2}$ particles is placed in an external magnetic field H. The behavior of the entropy of the system as a function of energy is given by

(a)



(b)



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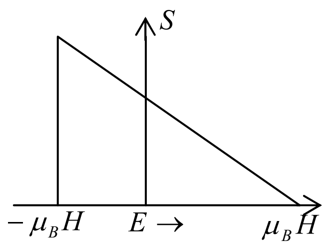

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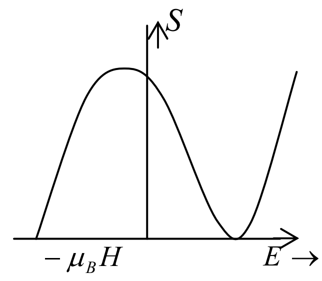

Ans. : (a)

Solution: $\frac{S}{Nk}=\frac{-N\varepsilon+U}{2N\varepsilon}\ln\left(\frac{N\varepsilon+U}{2\varepsilon}\right)-\frac{N\varepsilon-U}{2N\varepsilon}\ln\left(\frac{N\varepsilon-U}{2N\varepsilon}\right)$, where $\varepsilon=\mu H$. S is symmetrical about E.

Q10. A gas of N non-interacting particles is in thermal equilibrium at temperature T. Each particle can be in any of the possible non-degenerate states of energy $0,\ 2\varepsilon$ and 4ε. The average energy per particle of the gas, when $\beta\varepsilon<<1$, is

(a) 2ε (b) 3ε (c) $2\varepsilon/3$ (d) ε

Ans. : (a)

Solution: $E_1=0,\ E_2=2\varepsilon, E_3=4\varepsilon$, $Z=e^{-0\beta}+e^{-2\varepsilon\beta}+e^{-4\varepsilon\beta}$ $\Rightarrow\langle E\rangle=\frac{0\cdot e^{-o\beta}+2\varepsilon e^{-2\varepsilon\beta}+4\varepsilon e^{-4\varepsilon\beta}}{e^{-0\beta}+e^{-2\varepsilon\beta}+e^{-4\varepsilon 1\beta}}$

$$\Rightarrow\langle E\rangle=\frac{2\varepsilon e^{-2\varepsilon\beta}+4\varepsilon e^{-4\varepsilon\beta}}{1+e^{-2\varepsilon\beta}+e^{-4\varepsilon\beta}}=\frac{2\varepsilon(1-2\varepsilon\beta....)+4\varepsilon(1-(4\varepsilon\beta)....)}{1+(1-2\varepsilon\beta....)+(1-4\varepsilon\beta.....)}=\frac{2\varepsilon+4\varepsilon}{1+1+1}=\frac{6\varepsilon}{3}=2\varepsilon$$

where $\beta\varepsilon<<1$.

Q11. A one-dimensional chain consists of a set of N rods each of length a. When stretched by a load, each rod can align either parallel or perpendicular to the length of the chain. The energy of a rod is $-\varepsilon$ when perpendicular to it. When the chain is in thermal equilibrium at temperature T, its average length is

(a) $Na/2$ (b) Na (c) $Na/\left(1+e^{-2\varepsilon/k_BT}\right)$ (d) $Na\left(1+e^{-2\varepsilon/k_BT}\right)$

Ans. : (c)

Solution: Let n_1 no. of rods are parallel and n_2 no. of rods are perpendicular.

Energy of rod when it is perpendicular $=-\varepsilon$

Energy of rod when it is parallel is ε.

$$P(-\varepsilon)=\frac{e^{-\beta(-\varepsilon)}}{e^{-\beta(-\varepsilon)}+e^{-\beta\varepsilon}}=\frac{e^{\beta\varepsilon}}{e^{\beta\varepsilon}+e^{-\beta\varepsilon}} \text{ and } P(\varepsilon)=\frac{e^{-\beta\varepsilon}}{e^{\beta\varepsilon}+e^{-\beta\varepsilon}}$$

Average length $=n_1aP(\varepsilon)+n_2aP(-\varepsilon)=\frac{n_1ae^{-\beta\varepsilon}+n_2ae^{\beta\varepsilon}}{e^{\beta\varepsilon}+e^{-\beta\varepsilon}}=\frac{Nae^{\beta\varepsilon}}{e^{\beta\varepsilon}+e^{-\beta\varepsilon}}=\frac{Na}{1+e^{-2\beta\varepsilon}}$

Since $P(-\varepsilon)>>P(\varepsilon)$ so $n_2\cong N$, $n_1\cong 0$.

Q12. The excitations of a three-dimensional solid are bosonic in nature with their frequency ω and wave-number k are related by $\omega \propto k^2$ in the large wavelength limit. If the chemical potential is zero, the behavior of the specific heat of the system at low temperature is proportional to

(a) $T^{1/2}$ (b) T (c) $T^{3/2}$ (d) T^3

Ans. : (c)

Solution: If dispersion relation is $\omega \propto k^s$,

At low temperature specific heat $\propto T^{3/s}$

Q13. Gas molecules of mass m are confined in a cylinder of radius R and height L (with $R >> L$) kept vertically in the Earth's gravitational field. The average energy of the gas at low temperatures (such that $mgL >> k_BT$) is given by

(a) $Nk_BT/2$ (b) $3Nk_BT/2$ (c) $2Nk_BT$ (d) $5Nk_BT/2$

Ans. : (d)

Solution: $Z = \frac{1}{h^3}\int e^{-\beta H} dp_x dp_y dp_z dxdydz$

$$Z = \int_{-\infty}^{\infty} e^{\frac{-p_x^2}{2mk_BT}} dp_x \int_{-\infty}^{\infty} e^{\frac{-p_y^2}{2mk_BT}} dp_y \int_{-\infty}^{\infty} e^{\frac{-p_z^2}{2mk_BT}} dp_z \int dx \quad dy \int_0^L e^{-\frac{mgz}{k_BT}} dz$$

$$Z = \pi R^2 \left(\frac{mk_BT}{2\pi\hbar^2}\right)^{\frac{3}{2}} \int_0^L e^{-\frac{mgz}{k_BT}} dz \Rightarrow Z = \pi R^2 \left(\frac{mk_BT}{2\pi\hbar^2}\right)^{\frac{3}{2}} \left(\frac{1-e^{-\frac{mgL}{k_BT}}}{\frac{mg}{k_BT}}\right)$$

$Z_N = Z^N$,

$$\Rightarrow \langle E \rangle = k_BT^2 \frac{\partial \ln z}{\partial T} = \frac{5Nk_BT}{2}, \text{ since } mgL >> k_BT$$

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Q14. Consider a system of non-interacting particles in d dimensional obeying the dispersion relation $\varepsilon = Ak^s$, where ε is the energy, k is the wave vector; s is an integer and A is constant. The density of states, $N(\varepsilon)$, is proportional to

(a) $\varepsilon^{\frac{s}{d}-1}$ (b) $\varepsilon^{\frac{d}{s}-1}$ (c) $\varepsilon^{\frac{d}{s}+1}$ (d) $\varepsilon^{\frac{s}{d}+1}$

Ans. : (b)

Solution: We can solve this problem with intuition for example $\varepsilon = Ak^2$

Density of state in 3-dimensional $N(\varepsilon) \propto \varepsilon^{\frac{1}{2}} = \varepsilon^{\frac{3}{2}-1}$

Density of state in 2-dimensional $N(\varepsilon) \propto \varepsilon^{0} = \varepsilon^{\frac{2}{2}-1}$

Density of state in 1-dimensional $N(\varepsilon) \propto \varepsilon^{\frac{-1}{2}} = \varepsilon^{\frac{1}{2}-1}$

Density of state in d-dimensional, where $\varepsilon = Ak^s \Rightarrow N(\varepsilon) \propto \varepsilon^{\frac{d}{s}-1}$

Q15. The number of ways in which N identical bosons can be distributed in two energy levels, is

(a) $N+1$ (b) $\frac{N(N-1)}{2}$ (c) $\frac{N(N+1)}{2}$ (d) N

Ans. : (a)

Solution: Number of boson $= N$, Number of energy level $= g$

So number of ways to distribute N boson into g level is, $W = {}^{N+g-1}c_N = N+1$ since $g = 2$.

Q16. The free energy of the gas of N particles in a volume V and at a temperature T is $F = Nk_BT\ln\left[a_0V(k_BT)^{5/2}/N\right]$, where a_0 is a constant and k_B denotes the Boltzmann constant. The internal energy of the gas is

(a) $\frac{3}{2}Nk_BT$ (b) $\frac{5}{2}Nk_BT$

(c) $Nk_BT\ln\left[a_0V(k_BT)^{5/2}/N\right] - \frac{3}{2}Nk_BT$ (d) $Nk_BT\ln\left[a_0V/(k_BT)^{5/2}\right]$

Ans. : (b)

Solution: $F = Nk_BT\ln\left[a_0V(k_BT)^{5/2}/N\right]$, $F = U - TS$, $U = F + TS$

$$dF = -SdT - PdV \Rightarrow \left(\frac{\partial F}{\partial T}\right)_V = -S \text{ or } S = -\left(\frac{\partial F}{\partial T}\right)_V \Rightarrow U = F - T\left(\frac{\partial F}{\partial T}\right)_V$$

$$F = Nk_BT\ln\left(CT^{5/2}\right) \text{ where } C = \frac{a_0Vk_B^{5/2}}{N}$$

$$\left(\frac{\partial F}{\partial T}\right)_V = Nk_B\ln\left(CT^{5/2}\right) + Nk_BT\frac{C}{CT^{5/2}}\frac{5}{2}T^{3/2} \Rightarrow T\left(\frac{\partial F}{\partial T}\right)_V = Nk_BT\ln\left(CT^{5/2}\right) + \frac{5}{2}Nk_BT$$

$$T\left(\frac{\partial F}{\partial T}\right)_V = F + \frac{5}{2}Nk_BT \Rightarrow U = F - T\left(\frac{\partial F}{\partial T}\right)_V = -\frac{5}{2}Nk_BT.$$

Q17. A system has two normal modes of vibration, with frequencies ω_1 and $\omega_2 = 2\omega_1$. What is the probability that at temperature T, the system has an energy less than $4\hbar\omega_1$?

[In the following $x = e^{-\beta\hbar\omega_1}$ and Z is the partition function of the system.]

(a) $x^{3/2}\left(x + 2x^2\right)/Z$ (b) $x^{3/2}\left(1 + x + x^2\right)/Z$

(c) $x^{3/2}\left(1 + 2x^2\right)/Z$ (d) $x^{3/2}\left(1 + x + 2x^2\right)/Z$

Ans. : (d)

Solution: There is two normal mode so there is two degree of freedom.

Energy of harmonic oscillator is $E = \left(n_1 + \frac{1}{2}\right)\hbar\omega_1 + \left(n_2 + \frac{1}{2}\right)\hbar\omega_2$.

$$E = \left(n_1 + \frac{1}{2}\right)\hbar\omega_1 + \left(n_2 + \frac{1}{2}\right)\hbar 2\omega_1 \text{ where } n_1 = 0,1,2,3.... \text{ and } n_2 = 0,1,2,3....$$

Ground state energy $E = \frac{3\hbar\omega_1}{2}$, first excited state energy $E = \frac{5\hbar\omega_1}{2}$. Second excited state energy $E = \frac{7\hbar\omega_1}{2}$ which is doubly degenerate state so $g = 2$, other state have more energy than $4\hbar\omega_1$.

$$P(E < 4\hbar\omega_1) = \frac{e^{-\frac{3\beta\hbar\omega_1}{2}} + e^{-\frac{5\beta\hbar\omega_1}{2}} + 2e^{-\frac{7\beta\hbar\omega_1}{2}}}{Z} = \frac{x^{3/2}\left(1 + x + 2x^2\right)}{Z} \text{ where } x = e^{-\beta\hbar\omega_1}.$$

Q18. Bose condensation occurs in liquid He^4 kept at ambient pressure at $2.17\,K$. At which temperature will Bose condensation occur in He^4 in gaseous state, the density of which is 1000 times smaller than that of liquid He^4? (Assume that it is a perfect Bose gas.)

(a) $2.17\,mK$ (b) $21.7\,mK$ (c) $21.7\,\mu K$ (d) $2.17\,\mu K$

Ans. : (b)

Solution: For bosons $T \propto \left(\frac{N}{V}\right)^{\frac{2}{3}}$

Q19. Consider black body radiation contained in a cavity whose walls are at temperature T. The radiation is in equilibrium with the walls of the cavity. If the temperature of the walls is increased to $2T$ and the radiation is allowed to come to equilibrium at the new temperature, the entropy of the radiation increases by a factor of

(a) 2 (b) 4 (c) 8 (d) 16

Ans. : (c)

Solution: For Black Body, energy is given by $F = \frac{-8\pi^5 k_B^4 T^4}{45\hbar^2 C^3} V$, $S = -\left(\frac{\partial F}{\partial T}\right)_V = \left(\frac{32\pi^5 k_B^4}{45\hbar^3 C^3}\right) VT^3$.

$\Rightarrow S \propto T^3$, If temperate increase from T to $2T$ then entropy will incase S to $8S$.

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Q20. The entropy of a system, (S), is related to the accessible phase space volume Γ by $S = k_B \ln \Gamma(E, N, V)$ where E, N and V are the energy, number of particles and volume respectively. From this one can conclude that Γ

(a) does not change during evolution to equilibrium

(b) oscillates during evolution to equilibrium

(c) is a maximum at equilibrium

(d) is a minimum at equilibrium

Ans. : (c)

Solution: Entropy is maximum at equilibrium.

Q21. Let ΔW be the work done in a quasistatic reversible thermodynamic process. Which of the following statements about ΔW is correct?

(a) ΔW is a perfect differential if the process is isothermal

(b) ΔW is a perfect differential if the process is adiabatic

(c) ΔW is always a perfect differential

(d) ΔW cannot be a perfect differential

Ans. : (b)

Solution: Work done is perfect differential in adiabatic process.

Q22. The free energy difference between the superconducting and the normal states of a material is given by $\Delta F = f_S - f_N = \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4$ where ψ is an order parameter and α and β are constants s.t. $\alpha > 0$ in Normal and $\alpha < 0$ in the super conducting state, while $\beta > 0$ always, minimum value of ΔF is

(a) $-\frac{\alpha^2}{\beta}$ (b) $-\frac{\alpha^2}{2\beta}$ (c) $-\frac{3\alpha^2}{2\beta}$ (d) $-\frac{5\alpha^2}{2\beta}$

Ans. : (b)

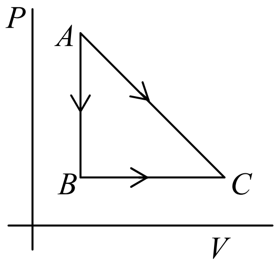
Solution: $\Delta F = \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 \Rightarrow \frac{\Delta F}{\partial \psi} = 2\alpha|\psi| + \frac{4\beta}{2}|\psi|^3$

$2\alpha|\psi| + 2\beta|\psi|^3 = 0 \Rightarrow |\psi|^2 = -\frac{\alpha}{\beta}$

Putting the value, $\Delta F = -\frac{\alpha^2}{\beta} + \frac{\beta}{2}\times\frac{\alpha^2}{\beta^2} \Rightarrow \Delta F\big|_{\min} = -\frac{\alpha^2}{2\beta}$

Q23. A given quantity of gas is taken from the state $A \to C$ reversibly, by two paths, $A \to C$ directly and $A \to B \to C$ as shown in the figure.

During the process $A \to C$ the work done by the gas is $100\,J$ and the heat absorbed is $150\,J$. If during the process $A \to B \to C$ the work done by the gas is $30\,J$, the heat absorbed is



(a) $20\,J$ (b) $80\,J$ (c) $220\,J$ (d) $280\,J$

Ans. : (b)

Solution: During path AC, $dU = dQ - dW = 150 - 100 = 50\ J$

Since, internal energy is point function, so dU will same in all path

In path ABC, $dQ = dU + dW = 50 + 30 = 80\ J$.

Q24. Consider a one-dimensional Ising model with N spins, at very low temperatures when almost all spins are aligned parallel to each other. There will be a few spin flips with each flip costing an energy $2J$. In a configuration with r spin flips, the energy of the system is $E = -NJ + 2rJ$ and the number of configuration is ${}^{N}C_r; r$ varies from 0 to N. The partition function is

(a) $\left(\frac{J}{k_B T}\right)^N$ (b) $e^{-NJ/k_B T}$ (c) $\left(\sinh\frac{J}{k_B T}\right)^N$ (d) $\left(\cosh\frac{J}{k_B T}\right)^N$

Ans. : (d)

Solution: Let us consider only three energy levels, $E_r = -2J + 2rJ$ i.e. $E_0 = -2J$, $E_1 = 0$ and $E_2 = 2J$.

$$Q_2 = \frac{\left({}^2C_0 e^{-\beta E_0} + {}^2C_1 e^{-\beta E_1} + {}^2C_2 e^{-\beta E_2}\right)}{\sum_{r=0}^{2} {}^2C_r} = \frac{\left(e^{\beta 2J} + 2e^0 + e^{\beta 2J}\right)}{4} = \frac{\left(e^{\beta J} + e^{\beta J}\right)^2}{4}$$

$$Q_2 = \left(\frac{e^{\beta J} + e^{\beta J}}{2}\right)^2 = (\cosh \beta J)^2 \Rightarrow (\cosh \beta J)^2 \Rightarrow Q_N = (\cosh \beta J)^N.$$

Q25. Consider a system of three spins S_1, S_2 and S_3 each of which can take values +1 and -1. The energy of the system is given by $E = -J[S_1S_2 + S_2S_3 + S_3S_1]$ where J is a positive constant. The minimum energy and the corresponding number of spin configuration are, respectively,

(a) J and 1 (b) $-3J$ and 1 (c) $-3J$ and 2 (d) $-6J$ and 2

Ans. : (c)

Solution: If we take $S_1 = S_2 = S_3 = +1$ i.e. $\underset{S_1}{\uparrow} \quad \underset{S_2}{\uparrow} \quad \underset{S_3}{\uparrow}$

Then energy, $E = -J[1\times1 + 1\times1 + 1\times1] = -3J$

Again $S_1 = S_2 = S_3 = -1$, then $\downarrow \quad \downarrow \quad \downarrow$

Energy $(E) = -3J$

So, minimum energy is $(-3J)$ and there are two spin configuration.

If we take $\underset{S_1}{\uparrow} \; \underset{S_2}{\downarrow} \; \underset{S_3}{\uparrow}$

Then we get Maximum energy $E = J$.

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Q26. Ten grams of ice at $0^0 C$ is added to a beaker containing 30 grams of water at $25^0 C$. What is the final temperature of the system when it comes to thermal equilibrium? (The specific heat of water is $1 cal/gm/^0 C$ and latent heat of melting of ice is $80 cal/gm$)

(a) $0^0 C$ (b) $7.5^0 C$ (c) $12.5^0 C$ (d) $-1.25^0 C$

Ans. : (a)

Solution: The amount of heat required to melt the ice of mass $10 gm$ at $0^0 C$ is

$Q = m \times L = 10 \times 80 = 800 Cal$, where L is the latent heat of melting of ice and m is the mass of the ice. The amount of heat available in water of mass $30 gm$ at $25^0 C$ is

$$Q = m \times C_v \times T = 30 \times 1 \times 25 = 750 Cal$$

Since the heat available is less than the heat required to melt the ice therefore ice will not melt as a result the temperature of the system will be at $0^0 C$ only.

Q27. A vessel has two compartments of volume V_1 and V_2, containing an ideal gas at pressures p_1 and p_2, and temperatures T_1 and T_2 respectively. If the wall separating the compartments is removed, the resulting equilibrium temperature will be

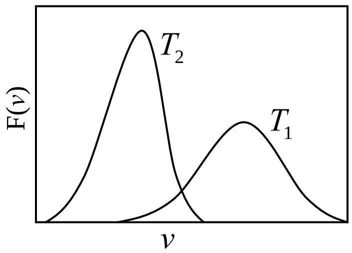
(a) $\dfrac{p_1 T_1 + p_2 T_2}{p_1 + p_2}$ (b) $\dfrac{V_1 T_1 + V_2 T_2}{V_1 + V_2}$ (c) $\dfrac{p_1 V_1 + p_2 V_2}{(p_1 V_1 / T_1) + (p_2 V_2 / T_2)}$ (d) $(T_1 T_2)^{1/2}$

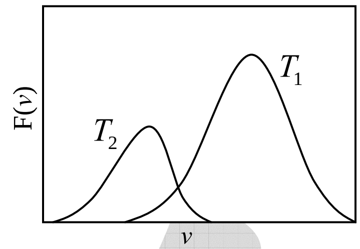
Ans. : (c)

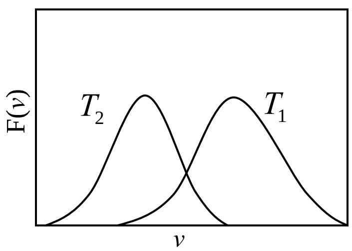
Solution: $V = V_1 + V_2$, $n = n_1 + n_2 = \dfrac{p_1 V_1}{T_1} + \dfrac{p_2 V_2}{T_2}$, $U_1 + U_2 = U$, $n_1 C_v T_1 + n_2 C_v T_2 = n C_v T$,

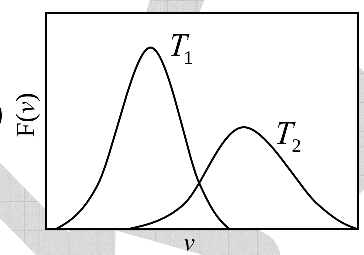
$$n_1 T_1 + n_2 T_2 = nT \Rightarrow T = \frac{p_1 V_1 + p_2 V_2}{\dfrac{p_1 V_1}{T_1} + \dfrac{p_2 V_2}{T_2}}$$

Q28. For temperature $T_1 > T_2$, the qualitative temperature dependence of the probability distribution $F(v)$ of the speed v of a molecule in three dimensions is correctly represented by the following figure:

(a) 

(b) 

(c) 

(d) 

Ans. : (a)

Solution: Area under the $F(v)$ is conserve and the mean velocity shift towards right for higher temperature.

Q29. A system of non-interacting spin-$1/2$ charged particles are placed in an external magnetic field. At low temperature T, the leading behavior of the excess energy above the ground state energy, depends on T as: (c is a constant)

(a) cT (b) cT^3 (c) $e^{-c/T}$ (d) c (is independent of T)

Ans. : (c)

Solution: $U = -\mu_B H \tanh\frac{\mu_B H}{kT} = -\mu_B H\left(\frac{e^{\frac{\mu_B H}{kT}} - e^{-\frac{\mu_B H}{kT}}}{e^{\frac{\mu_B H}{kT}} + e^{-\frac{\mu_B H}{kT}}}\right)$

Excess energy from the ground level

$$= -\mu_B H\left(\frac{e^{\frac{\mu_B H}{kT}} - e^{-\frac{\mu_B H}{kT}}}{e^{\frac{\mu_B H}{kT}} + e^{-\frac{\mu_B H}{kT}}}\right) - (-\mu_B H) = \mu_B H\left[1 - \left(\frac{e^{\frac{\mu_B H}{kT}} - e^{-\frac{\mu_B H}{kT}}}{e^{\frac{\mu_B H}{kT}} + e^{-\frac{\mu_B H}{kT}}}\right)\right] = \mu_B H\left(\frac{2e^{-\frac{\mu_B H}{kT}}}{e^{\frac{\mu_B H}{kT}} + e^{-\frac{\mu_B H}{kT}}}\right)$$

At low temperature, the lower value, $\Delta U \propto e^{-C/T}$, where $C = \mu_B H$.

Q30. Consider a system of two Ising spins S_1 and S_2 taking values ± 1 with interaction energy given by $\varepsilon = -JS_1S_2$, when it is in thermal equilibrium at temperature T. For large T, the average energy of the system varies as C/k_BT, with C given by

(a) $-2J^2$ (b) $-J^2$ (c) J^2 (d) $4J$

Ans. : (b)

Solution: The interaction energy is given by $E = -J\,S_1S_2$ where S_1 and S_2 taking values ± 1.

Possible values of the Energy of the system are

$E_1 = -J\,1\cdot 1 = -J$, $E_2 = -J(-1)\cdot(1) = +J$

$E_3 = -J(1)\cdot(-1) = +J$, $E_4 = -J(-1)\cdot(-1) = -J$

$$\langle U\rangle = \frac{\sum_r E_r g_r e^{-\frac{E_r}{kT}}}{\sum_r g_r e^{-\frac{E_r}{kT}}} = \frac{-2Je^{\frac{J}{kT}} + 2Je^{-\frac{J}{kT}}}{2e^{\frac{J}{kT}} + 2e^{-\frac{J}{kT}}} = -J\left(\frac{e^{\frac{J}{kT}} - e^{-\frac{J}{kT}}}{e^{\frac{J}{kT}} + e^{-\frac{J}{kT}}}\right) = -J\frac{\left(1+\frac{J}{kT} - \left(1-\frac{J}{kT}\right)\right)}{1+\left(\frac{J}{kT}\right)+1-\left(\frac{J}{kT}\right)}$$

$$\Rightarrow \langle U\rangle = -\frac{J^2}{kT} \Rightarrow C = -J^2 \quad \text{(For large } T,\ \frac{J}{kT} << 1)$$

Q31. Consider two different systems each with three identical non-interacting particles. Both have single particle states with energies $\varepsilon_0, 3\varepsilon_0$ and $5\varepsilon_0, (\varepsilon_0 > 0)$. One system is populated by spin $-\frac{1}{2}$ fermions and the other by bosons. What is the value of $E_F - E_B$ where E_F and E_B are the ground state energies of the fermionic and bosonic systems respectively?

(a) $6\varepsilon_0$ (b) $2\varepsilon_0$ (c) $4\varepsilon_0$ (d) ε_0

Ans. : (b)

Solution: Energy of Fermion = $2\times 1\varepsilon_0 + 3\varepsilon_0 = 5\varepsilon_0$

Energy of boson = $3\times 1\varepsilon_0 = 3\varepsilon_0$

$E_F - E_B = 5\varepsilon_0 - 3\varepsilon_0 = 2\varepsilon_0$

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Q32. Three identical spin-$\frac{1}{2}$ fermions are to be distributed in two non-degenerate distinct energy levels. The number of ways this can be done is

(a) 8 (b) 4 (c) 3 (d) 2

Ans. : (b)

Solution: Total number of degeneracy

$g =$ (Number of energy state (n))$\times$(Number of degeneracy due to spin ($2s+1$))

$$n = 2,\ s = \frac{1}{2}, \qquad g = 2\times(2.\frac{1}{2}+1) = 4$$

Number of particle, $N = 3$. So number of ways, ${}^{g}c_N = {}^{4}c_3 = 4$

Q33. Consider the melting transition of ice into water at constant pressure. Which of the following thermodynamic quantities **does not** exhibit a discontinuous change across the phase transition?

(a) Internal energy (b) Helmholtz free energy (c) Gibbs free energy (d) Entropy

Ans. : (c)

Solution: Ice to water: 1st order phase transition.

So Gibbs free energy is continuous, so it doesn't exhibit discontinuous change.

Q34. Two different thermodynamic systems are described by the following equations of state: $\frac{1}{T^{(1)}} = \frac{3RN^{(1)}}{2U^{(1)}}$ and $\frac{1}{T^{(2)}} = \frac{5RN^{(2)}}{2U^{(2)}}$ where $T^{(1,2)}$, $N^{(1,2)}$ and $U^{(1,2)}$ are respectively, the temperatures, the mole numbers and the internal energies of the two systems, and R is the gas constant. Let U_{tot} denote the total energy when these two systems are put in contact and attain thermal equilibrium. The ratio $\frac{U^{(1)}}{U_{tot}}$ is

(a) $\frac{5N^{(2)}}{3N^{(1)}+5N^{(2)}}$ (b) $\frac{3N^{(1)}}{3N^{(1)}+5N^{(2)}}$ (c) $\frac{N^{(1)}}{N^{(1)}+N^{(2)}}$ (d) $\frac{N^{(2)}}{N^{(1)}+N^{(2)}}$

Ans. : (b)

Solution: $\frac{1}{T^{(1)}} = \frac{3RN^{(1)}}{2U^{(1)}}$ and $\frac{1}{T^{(2)}} = \frac{5RN^{(2)}}{2U^{(2)}}$

Now $U_{tot} = U^{(1)} + U^{(2)} = \frac{3}{2}RN^{(1)}T^{(1)} + \frac{5}{2}RN^{(2)}T^{(2)}$

$$\Rightarrow \frac{U^{(1)}}{U_{tot}} = \frac{\frac{1}{2}\left[3RN^{(1)}T^{(1)}\right]}{\frac{1}{2}\left[3N^{(1)}T^{(1)} + 5RN^{(2)}T^{(2)}\right]} = \frac{3N^{(1)}T^{(1)}}{3N^{(1)}T^{(1)} + 5N^{(2)}T^{(2)}}$$

At thermal equilibrium $T^{(1)} = T^{(2)}$, thus $\frac{3N^{(1)}}{3N^{(1)} + 5N^{(2)}}$

Q35. The speed v of the molecules of mass m of an ideal gas obeys Maxwell's velocity distribution law at an equilibrium temperature T. Let (v_x, v_y, v_z) denote the components of the velocity and k_B the Boltzmann constant. The average value of $(\alpha v_x - \beta v_y)^2$, where α and β are constants, is

(a) $(\alpha^2 - \beta^2)k_B T/m$ (b) $(\alpha^2 + \beta^2)k_B T/m$

(c) $(\alpha + \beta)^2 k_B T/m$ (d) $(\alpha - \beta)^2 k_B T/m$

Ans. : (b)

Solution: Ideal gas obeys Maxwell velocity distribution law at equilibrium temperature. Then average value of $(\alpha v_x - \beta v_y)^2$

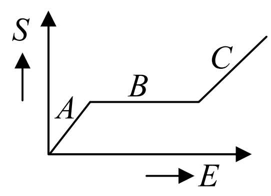
Now $\left\langle(\alpha v_x - \beta v_y)^2\right\rangle = \left\langle\alpha^2 v_x^2\right\rangle + \left\langle\beta^2 v_y^2\right\rangle - 2\left\langle\alpha\beta v_x v_y\right\rangle$

$\langle v_x\rangle = 0, \langle v_y\rangle = 0$ and $\left\langle v_x^2\right\rangle = \frac{k_B T}{m} = \left\langle v_y^2\right\rangle = \left\langle v_z^2\right\rangle$

Then $\left\langle(\alpha v_x - \beta v_y)^2\right\rangle = \alpha^2\left\langle v_x^2\right\rangle + \beta^2\left\langle v_y^2\right\rangle - 2\alpha\beta\langle v_x\rangle\langle v_y\rangle$

$\left\langle(\alpha v_x - \beta v_y)^2\right\rangle = \alpha^2\frac{k_B T}{m} + \beta^2\frac{k_B T}{m} = (\alpha^2 + \beta^2)\frac{kT}{m}$

Q36. The entropy S of a thermodynamic system as a function of energy E is given by the following graph. The temperatures of the phases A, B and C denoted by T_A, T_B and T_C respectively.

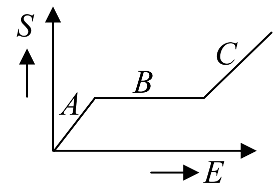


Satisfy the following inequalities:

(a) $T_C > T_B > T_A$ (b) $T_A > T_C > T_B$ (c) $T_B > T_C > T_A$ (d) $T_B > T_A > T_C$

Ans. : (c)

Solution: Temperatures of phase are: T_A, T_B, T_C



Since, $\left(\frac{dS}{dE}\right) = \frac{1}{T}$

Hence, $\frac{dS}{dE}$ will be slope, then it will be zero for B - phase

So $T_B = \infty$ and in C and A phases, internal energy of C phase is more, so $T_C > T_A$

Now $T_B > T_C > T_A$

Q37. A system of N classical non-interacting particles, each of mass m, is at a temperature T and is confined by the external potential $V(r) = \frac{1}{2}Ar^2$ (where A is a constant) in three dimensions. The internal energy of the system is

(a) $3Nk_BT$ (b) $\frac{3}{2}Nk_BT$ (c) $N(2mA)^{3/2}k_BT$ (d) $N\sqrt{\frac{A}{m}}\ln\left(\frac{k_BT}{m}\right)$

Ans. : (a)

Solution: $V(r) = \frac{1}{2}Ar^2 = \frac{1}{2}A(x^2 + y^2 + z^2)$ it is harmonic oscillator.

So its partition function will be $z_N = \frac{1}{\lfloor N}\left(\frac{kT}{\hbar\omega}\right)^{3N}$

Internal energy, $U = kT^2\frac{\partial \ln Z_N}{\partial T} = 3NkT$

Q38. A Carnot cycle operates as a heat engine between two bodies of equal heat capacity until their temperatures become equal. If the initial temperatures of the bodies are T_1 and T_2, respectively and $T_1 > T_2$, then their common final temperature is

(a) T_1^2 / T_2 (b) T_2^2 / T_1 (c) $\sqrt{T_1T_2}$ (d) $\frac{1}{2}(T_1 + T_2)$

Ans. : (c)

Solution: For heat Carnot engine the change in entropy for source and sink

$$dS_1 = \int_{T_1}^{T_F}\frac{dT}{T} = \log\left(\frac{T_F}{T_1}\right) \text{and } dS_2 = \int_{T_2}^{T_F}\frac{dT}{T} = \log\left(\frac{T_F}{T_2}\right)$$

$$\Delta S = dS_1 + dS_2 = \log\frac{T_F}{T_1} + \log\frac{T_F}{T_2}.$$

Since, Carnot engine is reversible in nature, so $\log\frac{(T_F)^2}{T_1T_2} = 0 \Rightarrow T_F = \sqrt{T_1T_2}$

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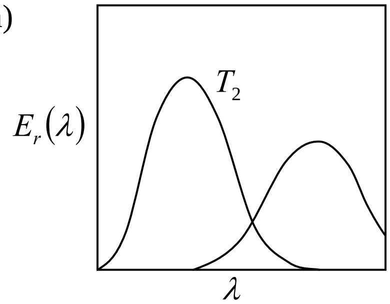
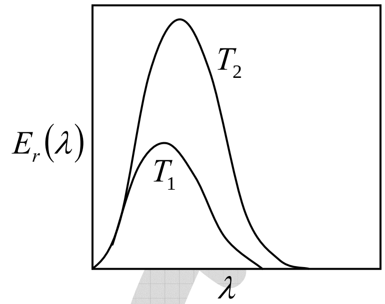
Q39. Which of the graphs below gives the correct qualitative behaviour of the energy density $E_r(\lambda)$ of blackbody radiation of wavelength λ at two temperatures T_1 and $T_2\,(T_1 < T_2)$?

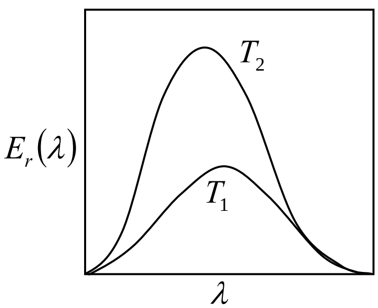
(a)



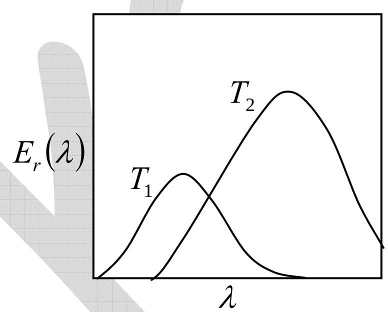
(b)



(c)



(d)



Ans. : (c)

Q40. A system can have three energy levels: $E = 0, \pm\varepsilon$. The level $E = 0$ is doubly degenerate, while the others are non-degenerate. The average energy at inverse temperature β is

(a) $-\varepsilon\tanh(\beta\varepsilon)$ (b) $\dfrac{\varepsilon\left(e^{\beta\varepsilon} - e^{-\beta\varepsilon}\right)}{\left(1 + e^{\beta\varepsilon} + e^{-\beta\varepsilon}\right)}$ (c) 0 (d) $-\varepsilon\tanh\left(\dfrac{\beta\varepsilon}{2}\right)$

Ans. : (d)

Solution: $E = 0, \pm\varepsilon,\ \ E = 0$ doubly degenerate

$$z = \sum g_i e^{-\beta E_i} \Rightarrow 2\times e^{-\beta\times 0} + e^{-\beta\varepsilon} + e^{\beta\varepsilon}$$

$$z = 2 + e^{\beta\varepsilon} + e^{-\beta\varepsilon} \Rightarrow \ln z = \ln\left(2 + e^{\beta\varepsilon} + e^{-\beta\varepsilon}\right)$$

$$\text{Now}\ \langle E\rangle = -\frac{\partial}{\partial\beta}\ln(z) = -\frac{\partial}{\partial\beta}\ln\left(2 + e^{\beta\varepsilon} + e^{-\beta\varepsilon}\right) = -\left[\frac{1}{2 + e^{\beta\varepsilon} + e^{-\beta\varepsilon}}\times\left(\varepsilon e^{\beta\varepsilon} - \varepsilon e^{-\beta\varepsilon}\right)\right]$$

$$\langle E\rangle = -\varepsilon\left[\frac{e^{\beta\varepsilon} - e^{-\beta\varepsilon}}{\left(e^{\frac{\beta\varepsilon}{2}} + e^{-\frac{\beta\varepsilon}{2}}\right)^2}\right] = -\varepsilon\left[\frac{\left(e^{\frac{\beta\varepsilon}{2}} - e^{\frac{-\beta\varepsilon}{2}}\right)}{\left(e^{\frac{\beta\varepsilon}{2}} + e^{\frac{-\beta\varepsilon}{2}}\right)}\right] = -\varepsilon\tanh\left(\frac{\beta\varepsilon}{2}\right)$$

Q41. The free energy F of a system depends on a thermodynamic variable ψ as

$$F = -a\psi^2 + b\psi^6$$

with $a, b > 0$. The value of ψ, when the system is in thermodynamic equilibrium, is

(a) zero (b) $\pm(a/6b)^{1/4}$ (c) $\pm(a/3b)^{1/4}$ (d) $\pm(a/b)^{1/4}$

Ans. : (c)

Solution: Frequency $F = -a\psi^2 + b\psi^6, \quad a,b>0$

F is equilibrium i.e. $\frac{\partial^2 F}{\partial\psi^2} > 0$, now $\frac{\partial F}{\partial\psi} = -2a\psi + 6b\psi^5$

$$\frac{\partial F}{\partial\psi} = 0 \Rightarrow 2a\psi = 6b\psi^5 \Rightarrow \frac{a}{3b} = \psi^4 \Rightarrow \psi = \pm\left(\frac{a}{3b}\right)^{1/4}$$

Q42. For a particular thermodynamic system the entropy S is related to the internal energy U and volume V by

$$S = cU^{3/4}V^{1/4}$$

where c is a constant. The Gibbs potential $G = U - TS + PV$ for this system is

(a) $\frac{3PU}{4T}$ (b) $\frac{cU}{3}$ (c) zero (d) $\frac{US}{4V}$

Ans. : (c)

Solution: $S = cU^{3/4}V^{1/4},\ dU = TdS - PdV$

$$\left(\frac{\partial S}{\partial U}\right)_V = \frac{1}{T} \Rightarrow \left(\frac{\partial S}{\partial U}\right)_V = \frac{1}{T} = \frac{c\times 3}{4}U^{-1/4}V^{1/4} \quad \Rightarrow T = \frac{4}{3c}\frac{U^{1/4}}{V^{1/4}}$$

$$\left(\frac{\partial U}{\partial V}\right)_S = -P = -\frac{S}{c}\frac{V^{-5/4}U^{\frac{1}{4}}}{3} \Rightarrow P = \frac{S}{c}\frac{V^{-5/4}}{3}U^{1/4}$$

$$G = U - \frac{4}{3c}\frac{U^{1/4}}{V^{1/4}}\times cU^{3/4}V^{1/4} + \frac{S}{c}\frac{V^{-5/4}}{3}U^{1/4}\times V = U - \frac{4}{3}U + \frac{1}{3}U = 0$$

Q43. The pressure of a non-relativistic free Fermi gas in three-dimensions depends, at $T = 0$, on the density of fermions n as

(a) $n^{5/3}$ (b) $n^{1/3}$ (c) $n^{2/3}$ (d) $n^{4/3}$

Ans. : (a)

Solution: Pressure $P = \frac{2}{3} n E_F$, $E_F \propto n^{2/3}$, at $T = 0$

$$P = \frac{2}{3} n \times n^{2/3} = \frac{2}{3} n^{5/3} \Rightarrow P \propto n^{5/3}$$

Q44. The vander Waals' equation of state for a gas is given by

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT$$

where P, V and T represent the pressure, volume and temperature respectively, and a and b are constant parameters. At the critical point, where all the roots of the above cubic equation are degenerate, the volume is given by

(a) $\frac{a}{9b}$ (b) $\frac{a}{27b^2}$ (c) $\frac{8a}{27bR}$ (d) $3b$

Ans. : (d)

Solution: $\left(P + \frac{a}{V^2}\right)(V - b) = RT$, for critical volume $\left(\frac{\partial P}{\partial V}\right) = 0$, $\left(\frac{\partial^2 P}{\partial V^2}\right) = 0$

$$PV + \frac{a}{V} - Pb - \frac{ab}{V^2} = RT$$

$$\frac{\partial P}{\partial V} = 0 \Rightarrow P - \frac{a}{V^2} + \frac{2ab}{V^3} = 0, \frac{\partial^2 P}{\partial V^2} = 0 \Rightarrow \frac{2a}{V^3} - \frac{6ab}{V^4} = 0 \Rightarrow \frac{2a}{V^3} = \frac{6ab}{V^4} \Rightarrow V_c = 3b$$

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Q45. The pressure P of a fluid is related to its number density ρ by the equation of state

$$P = a\rho + b\rho^2$$

where a and b are constants. If the initial volume of the fluid is V_0, the work done on the system when it is compressed, so as to increase the number density from an initial value of ρ_0 to $2\rho_0$ is

(a) $a\rho_0 V_0$ (b) $(a + b\rho_0)\rho_0 V_0$

(c) $\left(\frac{3a}{2} + \frac{7\rho_0 b}{3}\right)\rho_0 V_0$ (d) $(a \ln 2 + b\rho_0)\rho_0 V_0$

Ans. : (d)

Solution: $P = a\rho + b\rho^2 \Rightarrow P = a\frac{n}{V} + b\frac{n^2}{V^2}$ $\because \rho = \frac{n}{V}$

$$W = \int P \cdot dV = an\int_{V_1}^{V_2} \frac{dV}{V} + bn^2 \int_{V_1}^{V_2} \frac{dV}{V^2}, \quad \text{where } V_1 = \frac{n}{\rho_0}, V_2 = \frac{n}{2\rho_0}$$

$$\Rightarrow W = -n(a \ln 2 + b\rho_0) = -\rho_0 V_0 (a \ln 2 + b\rho_0), \quad \because n = \rho_0 V_0$$

Work done on the system $= -W = (a \ln 2 + b\rho_0)\rho_0 V_0$

Q46. An ideal Bose gas is confined inside a container that is connected to a particle reservoir. Each particle can occupy a discrete set of single-particle quantum states. If the probability that a particular quantum state is unoccupied is 0.1, then the average number of bosons in that state is

(a) 8 (b) 9 (c) 10 (d) 11

Ans. : (b)

Q47. In low density oxygen gas at low temperature, only the translational and rotational modes of the molecules are excited. The specific heat per molecule of the gas is

(a) $\frac{1}{2}k_B$ (b) k_B (c) $\frac{3}{2}k_B$ (d) $\frac{5}{2}k_B$

Ans. : (d)

Solution: Total D.O.F. = 3 transition + 2 rotation i.e. $f = 5$

$$U = f \cdot \frac{k_B T}{2} = \frac{5k_B T}{2} \Rightarrow C_V = \frac{\partial U}{\partial T} = \frac{5}{2}k_B$$

Q48. When a gas expands adiabatically from volume V_1 to V_2 by a quasi-static reversible process, it cools from temperature T_1 to T_2. If now the same process is carried out adiabatically and irreversibly, and T_2' is the temperature of the gas when it has equilibrated, then

(a) $T_2' = T_2$ (b) $T_2' > T_2$ (c) $T_2' = T_2\left(\frac{V_2 - V_1}{V_2}\right)$ (d) $T_2' = \frac{T_2 V_1}{V_2}$

Ans. : (b)

Q49. A random walker takes a step of unit length in the positive direction with probability $2/3$ and a step of unit length in the negative direction with probability $1/3$. The mean displacement of the walker after n steps is

(a) $n/3$ (b) $n/8$ (c) $2n/3$ (d) 0

Ans. : (a)

Solution: $P(+1) = \frac{2}{3} \Rightarrow P(-1) = \frac{1}{3}$

For one step $= +1 \times \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$, for n step $= \frac{n}{3}$

Q50. A collection N of non-interacting spins S_i, $i = 1, 2, \ldots, N$, $(S_i = \pm 1)$ is kept in an external magnetic field B at a temperature T. The Hamiltonian of the system is $H = -\mu B \Sigma_i S_i$. What should be the minimum value of $\frac{\mu B}{k_B T}$ for which the mean value $\langle S_i \rangle \geq \frac{1}{3}$?

(a) $\frac{1}{2} N \ln 2$ (b) $2\ln 2$ (c) $\frac{1}{2}\ln 2$ (d) $N \ln 2$

Ans. : (c)

Solution: $P(S_i = +1) = \frac{e^{-\frac{\mu B}{kT}}}{e^{-\frac{\mu B}{kT}} + e^{\frac{\mu B}{kT}}}$, $P(S_i = -1) = \frac{e^{\frac{\mu B}{kT}}}{e^{-\frac{\mu B}{kT}} + e^{\frac{\mu B}{kT}}}$

$$\langle S_i \rangle = \frac{+1 e^{-\frac{\mu B}{kT}} - e^{+\frac{\mu B}{kT}}}{e^{-\frac{\mu B}{kT}} + e^{\frac{\mu B}{kT}}} \Rightarrow \langle S_i \rangle = -\left(\tanh \frac{\mu B}{kT}\right)$$

For N particle $\langle S_i \rangle = -N \tanh \frac{\mu B}{kT}$

According to question, $\frac{\langle S_i \rangle}{N} \geq \frac{1}{3} \Rightarrow -\tanh\left(\frac{\mu B}{kT}\right) = \frac{1}{3} \Rightarrow \frac{\mu B}{kT} = \frac{1}{2}\ln 2$

NET/JRF (JUNE-2015)

Q51. A system of N non-interacting classical particles, each of mass m is in a two dimensional harmonic potential of the form $V(r)=\alpha(x^2+y^2)$ where α is a positive constant. The canonical partition function of the system at temperature T is $\left(\beta=\frac{1}{k_BT}\right)$:

(a) $\left[\left(\frac{\alpha}{2m}\right)^2\frac{\pi}{\beta}\right]^N$ (b) $\left(\frac{2m\pi}{\alpha\beta}\right)^{2N}$ (c) $\left(\frac{\alpha\pi}{2m\beta}\right)^N$ (d) $\left(\frac{2m\pi^2}{\alpha\beta^2}\right)^N$

Ans. (d)

Solution: $V(r)=\alpha(x^2+y^2)$

$$z_1=\frac{1}{h^2}\int_{-\infty}^{+\infty}e^{\frac{-p_x^2}{2mkT}}dp_x\int_{-\infty}^{+\infty}e^{\frac{-p_y^2}{2mkT}}dp_y\int_{-\infty}^{+\infty}e^{\frac{-\alpha x^2}{kT}}dx\int_{-\infty}^{+\infty}e^{\frac{-\alpha y^2}{kT}}dy$$

$$\Rightarrow z_1=\sqrt{\frac{2\pi mkT}{h^2}}\sqrt{\frac{2\pi mkT}{h^2}}2\times\frac{1}{2}\frac{1}{\sqrt{\frac{\alpha}{kT}}}\sqrt{\pi}2\times\frac{1}{2}\frac{\sqrt{\pi}}{\sqrt{\frac{\alpha}{kT}}}$$

$$z_1=\left(\frac{2\pi^2m}{h^2\alpha}\right)(kT)^2\Rightarrow z_N=\left(\frac{2\pi^2m}{h^2\alpha\beta^2}\right)^N$$

Q52. A system of N distinguishable particles, each of which can be in one of the two energy levels 0 and $\in$, has a total energy $n\in$, where n is an integer. The entropy of the system is proportional to

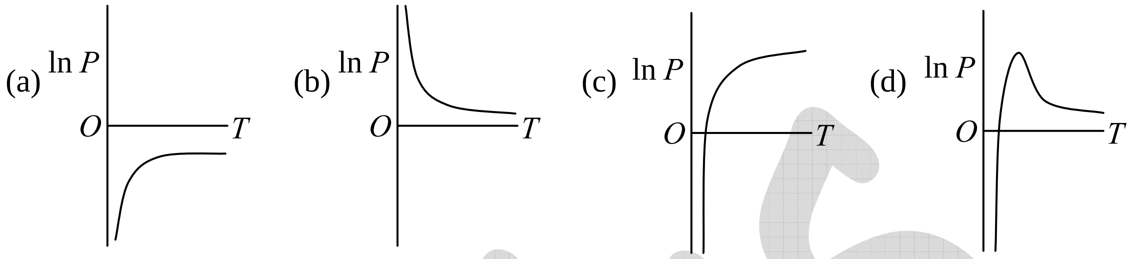
(a) $N\ln n$ (b) $n\ln N$ (c) $\ln\left(\frac{N!}{n!}\right)$ (d) $\ln\left(\frac{N!}{n!(N-n)!}\right)$

Ans. : (d)

Solution: No of ways for above configuration is $= {}^NC_n$

$$\Rightarrow W=\frac{\lfloor N}{\lfloor n\,\lfloor N-n}\Rightarrow \text{Entropy}=k\ln\frac{\lfloor N}{\lfloor n\,\lfloor N-n}$$

Q53. The condition for the liquid and vapour phases of a fluid to be in equilibrium is given by the approximate equation $\frac{dP}{dT} \approx \frac{Q_1}{Tv_{vap}}$ (Clausius-Clayperon equation) where v_{vap} is the volume per particle in the vapour phase, and Q_1 is the latent heat, which may be taken to be a constant. If the vapour obeys ideal gas law, which of the following plots is correct?



Ans. (c)

Solution: $\frac{dP}{dT} = \frac{Q_l}{Tv_{ap}}$, $v_{ap} = \frac{RT}{P} \Rightarrow \frac{dP}{dT} = \frac{Q_l P}{RT^2} \Rightarrow \frac{dP}{P} = \frac{Q_l}{R}\int \frac{dT}{T^2} \Rightarrow \ln P = -\frac{C}{T} + \alpha$

Q54. Consider three Ising spins at the vertices of a triangle which interact with each other with a ferromagnetic Ising interaction of strength J. The partition function of the system at temperature T is given by $\left(\beta = \frac{1}{k_B T}\right)$:

(a) $2e^{3\beta J} + 6e^{-\beta J}$
(b) $2e^{-3\beta J} + 6e^{\beta J}$
(c) $2e^{3\beta J} + 6e^{-3\beta J} + 3e^{\beta J} + 3e^{-\beta J}$
(d) $(2\cosh \beta J)^3$

Ans. (b)

Solution: $H = J(S_1S_2 + S_1S_3 + S_2S_3)$

S_1	S_2	S_3	E
1	1	1	$3J$
1	1	−1	$-J$
1	−1	1	
−1	1	1	
−1	−1	1	$-J$
−1	1	−1	
1	−1	−1	
−1	−1	−1	$3J$

$\Rightarrow \quad z = 2e^{-3\beta J} + 6e^{\beta J}$

Q55. A large number N of Brownian particles in one dimension start their diffusive motion from the origin at time $t=0$. The diffusion coefficient is D. The number of particles crossing a point at a distance L from the origin, per unit time, depends on L and time t as

(a) $\frac{N}{\sqrt{4\pi Dt}}e^{\frac{-L^2}{(4Dt)}}$ (b) $\frac{NL}{\sqrt{4\pi Dt}}e^{\frac{-4Dt}{L^2}}$ (c) $\frac{N}{\sqrt{16\pi Dt^3}}e^{\frac{-L^2}{(4Dt)}}$ (d) $Ne^{\frac{-4Dt}{L^2}}$

Ans. (a)

Solution: From Einstein Smoluchowski theory

$$p(x)dx = \frac{dx}{\sqrt{4\pi Dt}}\exp\left(\frac{-x^2}{4Dt}\right)$$

Number of particle passing from point L at origin $=\frac{N}{\sqrt{4\pi Dt}}.\exp\left(\frac{-L^2}{4Dt}\right)$

Q56. An ideal Bose gas in d-dimensions obeys the dispersion relation $\in\left(\vec{k}\right)=Ak^s$, where A and s are constants. For Bose-Einstein condensation to occur, the occupancy of excited states

$$N_e = c\int_0^\infty \frac{\in^{\frac{(d-s)}{s}}}{\left(e^{\beta(\in-\mu)}-1\right)}d\in$$

where c is a constant, should remain finite even for $\mu=0$. This can happen if

(a) $\frac{d}{s}<\frac{1}{4}$ (b) $\frac{1}{4}<\frac{d}{s}<\frac{1}{2}$ (c) $\frac{d}{s}>1$ (d) $\frac{1}{2}<\frac{d}{s}<1$

Ans. (c)

Solution: $Ne = c\int_0^\infty \frac{\in^{\frac{(d-s)}{s}}}{e^{\beta(\in-\mu)}-1}d\in$

B.E. condensation is possible in 3-D

For materlistic particle $g(\in)\propto\in^{\frac{1}{2}} \Rightarrow \frac{d-s}{s}=\frac{1}{2}\Rightarrow\frac{d}{s}=\frac{3}{2}$

For massless particle $g(\in)\propto\in^2 \Rightarrow \frac{d-s}{s}=2\Rightarrow\frac{d}{s}=3$

In both cases $\frac{d}{s}>1$

NET/JRF (DEC-2015)

Q57. The heat capacity of the interior of a refrigerator is $4.2\ kJ/K$. The minimum work that must be done to lower the internal temperature from $18^o\ C$ to $17^o\ C$, when the outside temperature is $27^o\ C$ will be

(a) $2.20\ kJ$ (b) $0.80\ kJ$ (c) $0.30\ kJ$ (d) $0.14\ kJ$

Ans. : (b)

Q58. For a system of independent non interacting one-dimensional oscillators, the value of the free energy per oscillator, in the limit $T \to 0$, is

(a) $\frac{1}{2}\hbar\omega$ (b) $\hbar\omega$ (c) $\frac{3}{2}\hbar\omega$ (d) 0

Ans. : (a)

Solution: For the given system $Z_N = \left[2\sinh\frac{\hbar\omega}{2kT}\right]^{-N} \Rightarrow F = -kT\ln Z_N = NkT\ln\left[2\sinh\left(\frac{\hbar\omega}{2kT}\right)\right]$

$$= NkT\ln\left[\frac{2\left(e^{\frac{\hbar\omega}{2kT}} - e^{-\frac{\hbar\omega}{2kT}}\right)}{2}\right] = NkT\ln\left[e^{\frac{\hbar\omega}{2kT}}\left(1-e^{-\frac{\hbar\omega}{kT}}\right)\right] = NkT\ln e^{\frac{\hbar\omega}{2kT}} + NkT\ln\left(1-e^{-\frac{\hbar\omega}{kT}}\right)$$

$$\frac{F}{N} = \frac{\hbar\omega}{2} + kT\ln\left(1-e^{-\frac{\hbar\omega}{kT}}\right) = \frac{\hbar\omega}{2} + 0 = \frac{\hbar\omega}{2} \qquad (\because kT \to 0)$$

Q59. The partition function of a system of N Ising spins is $Z = \lambda_1^N + \lambda_2^N$ where λ_1 and λ_2 are functions of temperature, but are independent of N. If $\lambda_1 > \lambda_2$, the free energy per spin in the limit $N \to \infty$ is

(a) $-k_BT\ln\left(\frac{\lambda_1}{\lambda_2}\right)$ (b) $-k_BT\ln\lambda_2$ (c) $-k_BT\ln(\lambda_1\lambda_2)$ (d) $-k_BT\ln\lambda_1$

Ans. : (d)

Solution: $Z = \lambda_1^N + \lambda_2^N$, $F = -kT\ln\left(\lambda_1^N + \lambda_2^N\right)$, it is given $\lambda_1 >> \lambda_2$

$$\Rightarrow F = -kT\ln\left[\lambda_1^N\left(1+\left(\frac{\lambda_2}{\lambda_1}\right)^N\right)\right], \qquad \frac{\lambda_2}{\lambda_1} \approx 0$$

$$F = -kT\ln\lambda_1^N = -NkT\ln\lambda_1 \Rightarrow \frac{F}{N} = -kT\ln\lambda_1$$

Q60. The Hamiltonian of a system of N non interacting spin $\frac{1}{2}$ particles is $H = -\mu_0 B\sum_i S_i^z$, where $S_i^z = \pm 1$ are components of i^{th} spin along an external magnetic field B. At a temperature T such that $e^{\frac{\mu_0 B}{k_B T}} = 2$. the specific heat per particle is

(a) $\frac{16}{25}k_B$ (b) $\frac{8}{25}k_B \ln 2$ (c) $k_B(\ln 2)^2$ (d) $\frac{16}{25}k_B(\ln 2)^2$

Ans. : (d)

Solution: For the given system $E = -\mu_0 B \tanh\frac{\mu_0 B}{kT}$

$$C_V = \left(\frac{\mu_0 B}{kT}\right)^2 Nk\sec^2 h\frac{\mu_0 B}{kT}$$

$$\frac{C_V}{N} = \left(\frac{\mu_0 B}{kT}\right)^2 k\frac{4}{\left(e^{\frac{\mu_0 B}{kT}} + e^{-\frac{\mu_0 B}{kT}}\right)^2} \qquad \because e^{\frac{\mu_0 B}{kT}} = 2 \Rightarrow \frac{\mu_0 B}{kT} = \ln 2$$

$$\frac{C_V}{N} = \left(\frac{\mu_0 B}{kT}\right)^2 k\frac{4}{\left(2+\frac{1}{2}\right)^2} = \left(\frac{\mu_0 B}{kT}\right)^2 k\frac{16}{25} = (\ln 2)^2\frac{16k}{25} = \frac{16}{25}k(\ln 2)^2$$

Q61. An ensemble of non-interacting spin - $\frac{1}{2}$ particles is in contact with a heat bath at temperature T and is subjected to an external magnetic field. Each particle can be in one of the two quantum states of energies $\pm\epsilon_0$. If the mean energy per particle is $-\epsilon_0/2$, then the free energy per particle is

(a) $-2\epsilon_0\frac{\ln(4/\sqrt{3})}{\ln 3}$ (b) $-\epsilon_0\ln(3/2)$ (c) $-2\epsilon_0\ln 2$ (d) $-\epsilon_0\frac{\ln 2}{\ln 3}$

Ans. : (a)

Solution: For the given system, partition function, $Z_n = 2^N\cosh\frac{\epsilon_0}{kT}$

Mean energy per unit particle $= \left(-\frac{\epsilon_0}{2}\right) = -\epsilon_0\tanh\frac{\epsilon_0}{kT}$

put $\frac{\epsilon_0}{kT} = \alpha \Rightarrow \tanh\alpha = \frac{1}{2} \Rightarrow \frac{e^{\alpha}-e^{-\alpha}}{e^{\alpha}+e^{-\alpha}} = \frac{1}{2}$

$$\Rightarrow e^{2\alpha} = 3 \Rightarrow \ \alpha = \frac{1}{2}\ln 3 \Rightarrow \frac{\in_0}{kT} = \frac{1}{2}\ln 3 \Rightarrow kT = \frac{2\in_0}{\ln 3}$$

It is given, $\frac{F}{N} = -kT\ln\left(2\cosh\frac{\in_0}{kT}\right) = -kT\ln\left(2\cosh\alpha\right)\sqrt{b^2-4ac}$

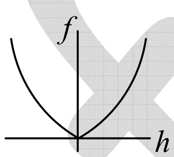
$$= -kT\ln\left(2\frac{e^{\alpha}+e^{-\alpha}}{2}\right) = -kT\ln\left[e^{-\alpha}\left(e^{2\alpha}+1\right)\right] = -kT\ln\left[e^{-\alpha}\left(3+1\right)\right] = -kT\ln\left[e^{-\alpha}4\right]$$

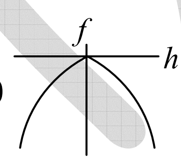
$$= -kT\left[\ln e^{-\alpha}+\ln 4\right] = kT\alpha - kT\ln 4 = kT\frac{\in_0}{kT} - kT\ln 4 = \in_0 - \frac{2\in_0}{\ln 3}\ln 4$$

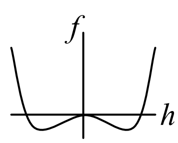
$$= \in_0\left[\frac{\ln 3 - 2\ln 4}{\ln 3}\right] = \in_0\left[\frac{\ln\left(\frac{3}{16}\right)}{\ln 3}\right] = -\in_0\left[\frac{\ln\left(\frac{16}{3}\right)}{\ln 3}\right]$$

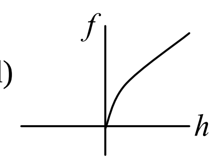
$$\Rightarrow \frac{F}{N} = -\in_0\left[\frac{\ln\left(\frac{4}{\sqrt{3}}\right)^2}{\ln 3}\right] = -2\in_0\frac{\ln\left(\frac{4}{\sqrt{3}}\right)}{\ln 3} = -2\in_0\frac{\ln\left(\frac{4}{\sqrt{3}}\right)}{\ln 3}$$

Q62. Which of the following graphs shows the qualitative dependence of the free energy $f\left(h,T\right)$ of a ferromagnet in an external magnetic field h, and at a fixed temperature $T < T_C$, where T_C is the critical temperature?

(a) 

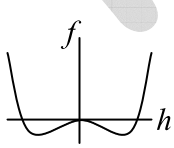
(b) 

(c) 

(d) 

Ans. : (c)

Solution: For super conductor state one will find two local minima



Option (c) is correct.

NET/JRF (JUNE-2016)

Q63. The specific heat per molecule of a gas of diatomic molecules at high temperatures is

(a) $8k_B$ (b) $3.5k_B$ (c) $4.5k_B$ (d) $3k_B$

Ans. : (b)

Solution: For high temperature all number are excited so degree of freedom for diatomic molecule is 7.

Internal energy is $\frac{fk_BT}{2}$, $U = \frac{7k_BT}{2}$, $C_V = \left(\frac{\partial U}{\partial T}\right)_V = 3.5k_B$

Q64. When an ideal monoatomic gas is expanded adiabatically from an initial volume V_0 to $3V_0$, its temperature changes from T_0 to T. Then the ratio $\frac{T}{T_0}$ is

(a) $\frac{1}{3}$ (b) $\left(\frac{1}{3}\right)^{\frac{2}{3}}$ (c) $\left(\frac{1}{3}\right)^{\frac{1}{3}}$ (d) 3

Ans. : (b)

Solution: For adiabatic process $PV^{\gamma} = k$, $T_0V_0^{\gamma-1} = k$

$$T_0V_0^{\gamma-1} = T(3V_0)^{\gamma-1} \Rightarrow T = T_0\left(\frac{V_0}{3V_0}\right)^{\gamma-1} \Rightarrow T = T_0\left(\frac{1}{3}\right)^{\nu-1}$$

For monoatomic gas $\gamma = \frac{5}{3}$

$$T = T_0\left(\frac{1}{3}\right)^{\frac{5}{3}-1} = T_0\left(\frac{1}{3}\right)^{\frac{2}{3}} \Rightarrow \frac{T}{T_0} = \left(\frac{1}{3}\right)^{2/3}$$

Q65. A box of volume V containing N molecules of an ideal gas, is divided by a wall with a hole into two compartments. If the volume of the smaller compartment is $\frac{V}{3}$, the variance of the number of particles in it, is

(a) $\frac{N}{3}$ (b) $\frac{2N}{9}$ (c) $\sqrt{N}$ (d) $\frac{\sqrt{N}}{3}$

Ans. : (b)

Solution: Probability that one particle is in smaller compartment having volume $\frac{V}{3}$, so $p = \frac{1}{3}$

There are only two options either particle is in left half or right half, so for one particle distribution is Bernoulli for Bernoulli's distribution $\sigma^2 = p(1-p)$. For N particle distribution is

$$\sigma^2 = Np(1-p) = N\frac{1}{3}\times\left(1-\frac{1}{3}\right),\ \sigma^2 = \frac{2N}{9}$$

Q66. A gas of non-relativistic classical particles in one dimension is subjected to a potential $V(x) = \alpha|x|$ (where α is a constant). The partition function is $\left(\beta = \frac{1}{k_B T}\right)$

(a) $\sqrt{\frac{4m\pi}{\beta^3\alpha^2h^2}}$ (b) $\sqrt{\frac{2m\pi}{\beta^3\alpha^2h^2}}$ (c) $\sqrt{\frac{8m\pi}{\beta^3\alpha^2h^2}}$ (d) $\sqrt{\frac{3m\pi}{\beta^3\alpha^2h^2}}$

Ans. : (c)

Solution: $z = \frac{1}{h}\int_{-\infty}^{\infty} e^{-\frac{p_x^2}{2mkT}} dp_x \int_{-\infty}^{\infty} e^{-\frac{\alpha|x|}{kT}} dx = \frac{1}{h}(2\pi mkT)^{1/2}\int_{-\infty}^{\infty} e^{-\frac{\alpha|x|}{kT}} dx$

$$\Rightarrow z = \left(\frac{2\pi mkT}{h^2}\right)^{1/2}\int_{-\infty}^{\infty} e^{-\frac{\alpha|x|}{kT}} dx$$

$$\because \int_{-\infty}^{\infty} e^{-\frac{\alpha|x|}{kT}} dx = \int_{-\infty}^{0} e^{+\frac{\alpha x}{kT}} dx + \int_{0}^{\infty} e^{-\frac{\alpha x}{kT}} dx = \frac{kT}{\alpha} + \frac{kT}{\alpha} = \frac{2kT}{\alpha}$$

$$z = \left(\frac{2\pi mkT}{h^2}\right)^{1/2}\left(\frac{2kT}{\alpha}\right) = \left(\frac{8\pi m}{h^2\beta^3\alpha^2}\right)^{\frac{1}{2}}, \quad \text{put } \beta = \frac{1}{kT}$$

Q67. The internal energy $E(T)$ of a system at a fixed volume is found to depend on the temperature T as $E(T) = aT^2 + bT^4$. Then the entropy $S(T)$, as a function of temperature, is

(a) $\frac{1}{2}aT^2 + \frac{1}{4}bT^4$ (b) $2aT^2 + 4bT^4$ (c) $2aT + \frac{4}{3}bT^3$ (d) $2aT + 2bT^3$

Ans. : (c)

Solution: From first law of thermodynamics,

$TdS = dE + PdV$, $\quad dE = TdS - PdV$, it is given $dV = 0$

$dE = TdS \Rightarrow dS = \frac{1}{T}dE$

$E = aT^2 + bT^4 \Rightarrow dE = 2aTdT + 4bT^3dT$

$$dS = \frac{1}{T}\left(2aTdT + 4bT^3dT\right) = 2adT + 4bT^2dT = 2aT + \frac{4bT^3}{3}$$

Q68. Consider a gas of Cs atoms at a number density of 10^{12} atoms/cc. when the typical inter-particle distance is equal to the thermal de Broglie wavelength of the particles, the temperature of the gas is nearest to (Take the mass of a Cs atom to be 22.7×10^{-26} kg)

(a) 1×10^{-9} K (b) 7×10^{-5} K (c) 1×10^{-3} K (d) 2×10^{-8} K

Ans. : (d)

Solution: When de Broglie wavelength = thermal wavelength

$$g_{3/2}(z) = 2.61$$

$$(2\pi mkT)^{3/2} = \frac{N}{V}\frac{h^3}{2.61} \Rightarrow 2\pi mkT = \left(\frac{N}{V}\right)^{2/3}\frac{h^2}{(2.61)^{2/3}}$$

$$T = \frac{1}{2\pi mk}(n)^{2/3}\frac{h^2}{(2.61)^{2/3}} = \frac{1(10^{12})^{2/3}(6.6\times10^{-34})^2}{2\times3.14\times22.7\times10^{-26}\times1.38\times10^{-23}\times(2.61)^{2/3}}$$

$$= \frac{(6.6)^2\times10^8\times10^{-64}\times10^{49}}{6.28\times22.7\times1.38\times(2.61)^{2/3}} = \frac{(6.6)^2\times10^{-7}}{6.28\times22.7\times1.38\times(2.61)^{2/3}}$$

$$= \frac{0.221\times10^{-7}}{(2.61)^{2/3}} = \frac{0.221}{1.895}\times10^{-7} = 0.116\times10^{-7} = 1.16\times10^{-8}$$

NET/JRF (DEC-2016)

Q69. The partition function of a two-level system governed by the Hamiltonian

$$H = \begin{bmatrix} \gamma & -\delta \\ -\delta & -\gamma \end{bmatrix} \text{ is}$$

(a) $2\sinh\left(\beta\sqrt{\gamma^2+\delta^2}\right)$

(b) $2\cosh\left(\beta\sqrt{\gamma^2+\delta^2}\right)$

(c) $\frac{1}{2}\left[\cosh\left(\beta\sqrt{\gamma^2+\delta^2}\right)+\sinh\left(\beta\sqrt{\gamma^2+\delta^2}\right)\right]$

(d) $\frac{1}{2}\left[\cosh\left(\beta\sqrt{\gamma^2+\delta^2}\right)-\sinh\left(\beta\sqrt{\gamma^2+\delta^2}\right)\right]$

Ans. : (b)

Solution: $H = \begin{bmatrix} \gamma & -\delta \\ -\delta & -\gamma \end{bmatrix}$

The eigen value is given by $E_1 = +\sqrt{\gamma^2+\delta^2}$ and $E_2 = -\sqrt{\gamma^2+\delta^2}$

$$Z = \text{trace}\left(e^{-\beta H}\right) = e^{-\beta E_1} + e^{-\beta E_2} = e^{-\beta\sqrt{\gamma^2+\delta^2}} + e^{-\beta\left(-\sqrt{\gamma^2+\delta^2}\right)} = 2\cosh \beta\sqrt{\gamma^2+\delta^2}$$

Q70. Consider a gas of N classical particles in a two-dimensional square box of side L. If the total energy of the gas is E, the entropy (apart from an additive constant) is

(a) $Nk_B \ln\left(\frac{L^2E}{N}\right)$ (b) $Nk_B \ln\left(\frac{LE}{N}\right)$ (c) $2Nk_B \ln\left(\frac{L\sqrt{E}}{N}\right)$ (d) $L^2k_B \ln\left(\frac{E}{N}\right)$

Ans. : (c)

Solution: $Z_N = \frac{1}{\lfloor N}\left[\frac{2\pi mkTL^2}{h^2}\right]^N$ $\qquad kT = \frac{E}{N}$

$$Z_N = \frac{1}{\lfloor N}\left[\frac{L^2E}{N}\right]^N \qquad \text{Assume } \frac{2\pi m}{h^2} = 1$$

$$\ln Z = -\ln\lfloor N + N\ln\left(\frac{L^2E}{N}\right) = -N\ln N + N + N\ln\left(\frac{L^2E}{N}\right)$$

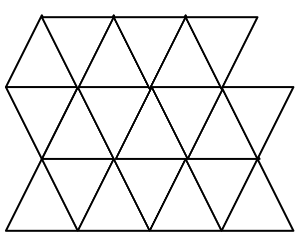
$$\therefore F = -kT\ln Z = NkT\ln N - NkT - NkT\ln\left(\frac{L^2E}{N}\right)$$

$$E = NkT$$

$$S = \frac{U-F}{T} = \frac{E-F}{T} = Nk - Nk\ln N + Nk + Nk\ln\left(\frac{L^2E}{N}\right)$$

$$= Nk\ln\left(\frac{L^2E}{N^2}\right) = Nk\ln\left(\frac{L\sqrt{E}}{N}\right)^2 = 2Nk\ln\left(\frac{L\sqrt{E}}{N}\right)$$

Q71. Consider a random walk on an infinite two-dimensional triangular lattice, a part of which is shown in the figure below.



If the probabilities of moving to any of the nearest neighbour sites are equal, what is the probability that the walker returns to the starting position at the end of exactly three steps?

(a) $\frac{1}{36}$ (b) $\frac{1}{216}$ (c) $\frac{1}{18}$ (d) $\frac{1}{12}$

Ans. : (c)

Solution: For walker to return to starting position it must move along an equivalent triangle in three steps.

For steps one any movement can result in equilateral triangle.

For step two, two out of six options will form equilateral triangle.

For step three, only one out of six options will form equilateral triangle

Total probability $=\frac{6}{6}\times\frac{2}{6}\times\frac{1}{6}=\frac{1}{18}$

Q72. An atom has a non-degenerate ground-state and a doubly-degenerate excited state. The energy difference between the two states is ε. The specific heat at very low temperatures $(\beta\varepsilon \gg 1)$ is given by

(a) $k_B(\beta\varepsilon)$ (b) $k_B e^{-\beta\varepsilon}$ (c) $2k_B(\beta\varepsilon)^2 e^{-\beta\varepsilon}$ (d) k_B

Ans. : (c)

Solution: Assume energy at ground state is 0 and energy at first excited state is $\in$. The partition function is $Z=1+2e^{-\beta\in}$

$$\text{Energy}=\frac{2\in e^{-\beta\in}}{\left(1+2e^{-\beta\in}\right)}$$

$$\text{Specific heat, } C_V=\left(\frac{\partial U}{\partial T}\right)_V=\frac{2\in e^{-\frac{\in}{kT}}(-\in)\frac{-1}{kT^2}}{\left(1+2e^{\frac{-\in}{kT}}\right)}+\frac{2\in e^{\frac{-\in}{kT}}\in\in\frac{2}{kT^2}}{\left(1+2e^{\frac{-\in}{kT}}\right)^2}$$

$$=2k\left(\frac{\in}{kT}\right)^2 e^{\frac{-\in}{kT}}\frac{\left(1+2e^{\frac{-\in}{kT}}\right)}{\left(1+2e^{\frac{-\in}{kT}}\right)^2}=2k(\beta\in)^2 e^{-\beta\in}\frac{\left(1+2e^{-\beta\in}\right)}{\left(1+2e^{-\beta\in}\right)^2}$$

$$C_V\simeq 2k(\beta\in)^2 e^{-\beta\in}, \qquad \beta\in\to\infty$$

Q73. The electrons in graphene can be thought of as a two-dimensional gas with a linear energy-momentum relation $E=|\vec{p}|v$, where $\vec{p}=(p_x,p_y)$ and v is a constant. If ρ is the number of electrons per unit area, the energy per unit area is proportional to

(a) $\rho^{3/2}$ (b) ρ (c) $\rho^{1/3}$ (d) ρ^2

Ans. : (a)

Solution: The number of k state in range k to $k+dk$:

In $2D$, it is given by $g(k)dk = \left(\frac{L}{2\pi}\right)^2 2\pi k dk$

Since, dispersion relation is $E = |P|v = \hbar k v$

$$g(E)dE = 2\times\left(\frac{L}{2\pi}\right)^2 2\pi \frac{EdE}{(\hbar v)^2} = \frac{L^2}{\pi\hbar^2 v^2}$$

The number of electron at $T = 0^0 K$ is

$$N = \int_0^{E_F} g(E)d(E) = \frac{L^2}{\pi\hbar^2 v^2}\int_0^{E_F} EdE = \frac{L^2}{2\pi\hbar^2 v^2}E_F^2 \Rightarrow 2\pi\hbar^2 v^2.\frac{N}{L^2} = E_F^2$$

$$E_F^2 = 2\pi\hbar^2 v^2 \rho \qquad \left(\rho = \frac{N}{L^2}\right)$$

The average energy at $T = 0K$ is

$$E_{av} = \frac{\int_0^{E_F} E\cdot g(E)dE}{N} = \frac{L^2}{N\pi\hbar^2 v^2}\int_0^{E_F} E^2 dE = \frac{L^2 E_F^3}{3N\pi\hbar^2 v^2}$$

$$E_{av} = \frac{L^2}{3N\pi\hbar^2 v^2}\times 2\pi\hbar^2 v^2 \rho\sqrt{2\pi\hbar^2 v^2}\,\rho^{1/2} = \frac{2L^2}{3N}\sqrt{2\pi}\hbar v \rho^{3/2}$$

$$\frac{E}{L^2} = \frac{NE_{av}}{L^2} = \frac{2}{3}\sqrt{2\pi}\hbar v\rho^{3/2} \qquad \Rightarrow \frac{E}{L^2} \propto \rho^{3/2}$$

NET/JRF (JUNE-2017)

Q74. A thermodynamic function

$$G(T,P,N) = U - TS + PV$$

is given in terms of the internal energy U, temperature T, entropy S, pressure P, volume V and the number of particles N. Which of the following relations is true? (In the following μ is the chemical potential.)

(a) $S = -\left.\frac{\partial G}{\partial T}\right|_{N,P}$ (b) $S = \left.\frac{\partial G}{\partial T}\right|_{N,P}$ (c) $V = -\left.\frac{\partial G}{\partial P}\right|_{N,T}$ (d) $\mu = -\left.\frac{\partial G}{\partial N}\right|_{P,T}$

Ans. : (a)

Solution: $G = U - TS + PV$

$dG = dU - Tds - sdT + PdV + VdP = TdS - PdV - TdS - SdT + PdV + VdP$

$dG = -SdT + VdP$

$\left(\frac{\partial G}{\partial T}\right)_{N,P} = -S$ and $\left.\frac{\partial G}{\partial P}\right|_{N,T} = V$

Q75. A box, separated by a movable wall, has two compartments filled by a monoatomic gas of $\frac{C_P}{C_V} = \gamma$. Initially the volumes of the two compartments are equal, but the pressures are $3P_0$ and P_0 respectively. When the wall is allowed to move, the final pressures in the two compartments become equal. The final pressure is

(a) $\left(\frac{2}{3}\right)^{\gamma} P_0$ (b) $3\left(\frac{2}{3}\right)^{\gamma} P_0$ (c) $\frac{1}{2}\left(1+3^{1/\gamma}\right)^{\gamma} P_0$ (d) $\left(\frac{3^{1/\gamma}}{1+3^{1/\gamma}}\right)^{\gamma} P_0$

Ans. : (c)

Solution: $V_1 + V_2 = 2V$, $V_2 = 2V - V_1$,

$3P_0V^{\gamma} = PV_1^{\gamma}$, $P_0V^{\gamma} = PV_2^{\gamma}$

$P_0V^{\gamma} = P\left(2V - V_1\right)^{\gamma}$

From (i) and (ii)

$$3 = \left(\frac{V_1}{2V - V_1}\right)^{\gamma} \Rightarrow 3^{1/\gamma} = \frac{V_1}{2V - V_1} \Rightarrow \frac{1}{3^{1/\gamma}} = \frac{2V - V_1}{V_1} = \frac{2V}{V_1} - 1 \Rightarrow V_1 = \frac{2V}{\left(1 + 1/3^{1/\gamma}\right)}$$

put the value of V_1 in (i)

$$3P_0V^{\gamma} = P\left(\frac{2V}{1 + 1/3^{1/\gamma}}\right)^{\gamma} \Rightarrow P = \frac{3P_0}{2^{\gamma}}\left(1 + 1/3^{1/\gamma}\right)^{\gamma} = \frac{P_0}{2^{\gamma}}\left(1 + 3^{1/\gamma}\right)^{\gamma}$$

Q76. A gas of photons inside a cavity of volume V is in equilibrium at temperature T. If the temperature of the cavity is changed to $2T$, the radiation pressure will change by a factor of

(a) 2 (b) 16 (c) 8 (d) 4

Ans. : (b)

Solution: For 3 dimensional system $P \propto T^4$

$$\frac{P_2}{P_1} = \left(\frac{2T}{T}\right)^4 \Rightarrow P_2 = P_1 2^4 = p_1 \times 16 = 16P_1$$

Q77. In a thermodynamic system in equilibrium, each molecule can exist in three possible states with probabilities $1/2, 1/3$ and $1/6$ respectively. The entropy per molecule is

(a) $k_B \ln 3$ (b) $\frac{1}{2}k_B \ln 2 + \frac{2}{3}k_B \ln 3$

(c) $\frac{2}{3}k_B \ln 2 + \frac{1}{2}k_B \ln 3$ (d) $\frac{1}{2}k_B \ln 2 + \frac{1}{6}k_B \ln 3$

Ans. : (c)

Solution: $S = -k_B \sum_i Pi \ln Pi$

$P_1 = \frac{1}{2}, \; P_2 = 1/3$ and $P_3 = 1/6 \cdot$

$$S = -k_B\left(\frac{1}{2}\ln 1/2 + 1/3 \ln 1/3 + 1/6 \ln 1/6\right)\cdot$$

$$= -k_b\left(\frac{1}{2}(\ln 1 - \ln 2) + \frac{1}{3}(\ln 1 - \ln 3) + \frac{1}{6}(\ln 1 - \ln 6)\right)$$

$$= k_B\left[\frac{1}{2}\ln 2 + \frac{1}{3}\ln 3 + \frac{1}{6}\ln 2 + \frac{1}{6}\ln 3\right] = k_B\left[\frac{1}{2}\ln 2 + \frac{1}{6}\ln 2 + \frac{1}{3}\ln 3 + \frac{1}{6}\ln 3\right]$$

$$S = k_B\left[\frac{3\ln 2 + \ln 2}{6} + \frac{2\ln 3 + \ln 3}{6}\right] = k_B\left(\frac{4\ln 2}{6} + \frac{3\ln 3}{6}\right) = k_B\left[\frac{2}{3}\ln 2 + \frac{1}{2}\ln 3\right]$$

Q78. The single particle energy levels of a non-interacting three-dimensional isotropic system, labelled by momentum k, are proportional to k^3. The ratio $\bar{P}/\in$ of the average pressure $\bar{P}$ to the energy density $\in$ at a fixed temperature, is

(a) $1/3$ (b) $2/3$ (c) 1 (d) 3

Ans. : (c)

Solution: $E \propto p^s$, where p is momentum

$P = \frac{s}{3}\left(\frac{E}{V}\right)$, where P is pressure

$\frac{P}{E} \propto \frac{s}{3}$.

In problem, $E \propto k^3$, so, $s = 3$

pressure $P = \frac{3}{3}\left(\frac{E}{V}\right) \Rightarrow P \propto \left(\frac{E}{V}\right)$ at fixed T.

Q79. The Hamiltonian for three Ising spins S_0, S_1 and S_2, taking values ± 1, is

$$H = -JS_0\left(S_1 + S_2\right)$$

If the system is in equilibrium at temperature T, the average energy of the system, in terms of $\beta = \left(k_B T\right)^{-1}$, is

(a) $-\dfrac{1+\cosh(2\beta J)}{2\beta \sinh(2\beta J)}$

(b) $-2J\left[1+\cosh(2\beta J)\right]$

(c) $-2/\beta$

(d) $-2J\dfrac{\sinh(2\beta J)}{1+\cosh(2\beta J)}$

Ans. : (d)

Solution: $H = -JS_0\left(S_1+S_2\right) = -J\left(S_0S_1 + S_0S_2\right)$ $\quad S_0 = \pm 1 \;\; S_1 = \pm S_2 = \pm 1$

S_0	S_1	S_2	E
1	1	1	$-2J$
-1	1	1	$2J$
1	-1	1	0
-1	-1	1	0
1	1	-1	0
-1	1	-1	0
1	-1	-1	$2J$
-1	-1	-1	$-2J$

$E_1 = -2J \qquad g_1 = 2$

$E_2 = 2J \qquad g_2 = 2$

$E_3 = 0 \qquad g_3 = 4$

$$U = \frac{\sum E_i g_i e^{-\beta E_i}}{\sum g_i e^{-\beta E_i}}$$

$$U = \frac{0 + 2J2e^{-2\beta J} + (-2J)2e^{2\beta J}}{4 + 2e^{-2\beta J} + 2e^{2\beta J}} = \frac{4J\left(e^{-2\beta J} - e^{2\beta J}\right)}{4 + 2\left(e^{-2\beta J} + e^{2\beta J}\right)} = \frac{(8J)(-\sinh 2\beta J)}{4 + 4\cosh 2\beta J}$$

$$\langle U \rangle = -\frac{2J\sinh 2\beta J}{1+\cosh 2\beta J}$$

NET/JRF (DEC - 2017)

Q80. A monoatomic gas ‘of volume V is in equilibrium in a uniform vertical cylinder, the lower end of which is closed by a rigid wall and the other by a frictionless piston. The piston is pressed lightly and released. Assume that the gas is a poor conductor of heat and the cylinder and piston are perfectly insulating. If the cross-sectional area of the cylinder is A, the angular frequency of small oscillations of the piston about the point of equilibrium, is

(a) $\sqrt{5gA/(3V)}$ (b) $\sqrt{4gA/(3V)}$ (c) $\frac{5}{3}\sqrt{gA/V}$ (d) $\sqrt{7gA/(5V)}$

Ans. : (a)

Solution: $PV^{\gamma}=(P+\Delta P)(V-\Delta V)^{\gamma} \Rightarrow PV^{\gamma}=P\left(1+\frac{\Delta P}{P}\right)V^{\gamma}\left(1-\frac{\Delta V}{V}\right)^{\gamma}$

$$\Rightarrow PV^{\gamma}=PV^{\gamma}\left(1+\frac{\Delta P}{P}\right)\left(1-\frac{\Delta V}{V}\right)^{\gamma} \Rightarrow \left(1+\frac{\Delta P}{P}\right)\left(1-\frac{\Delta V}{V}\right)^{\gamma}=1 \approx \left(1+\frac{\Delta P}{P}\right)\left(1-\frac{\gamma\Delta V}{V}\right)=1$$

$$1+\gamma\frac{\Delta V}{V}+\frac{\Delta P}{P}+\gamma\frac{\Delta P}{P}\frac{\Delta V}{V}=1 \qquad \text{(i)}$$

For small oscillation, also neglect $\frac{\Delta P}{P}\frac{\Delta V}{V}$

From equilibrium $P=\frac{F}{A}=\frac{mg}{A}$

From (i), we get $\frac{\Delta P}{P}=-\gamma\frac{\Delta V}{V}, \Delta P\cdot A=-\gamma\, mg.\Delta V, \Delta P\cdot A=-\gamma\frac{mg}{A}.\frac{A\,dx}{V}A.$

$$\Delta P\cdot A=-\gamma\frac{mgdx}{V}A \quad =F=-\frac{\gamma mg}{V}Adx$$

$$\omega=\sqrt{\frac{\gamma gA}{V}} \text{ and } \gamma=\frac{5}{3} \Rightarrow \omega=\sqrt{\frac{5}{3}\frac{gA}{V}}.$$

Q81. The relation between the internal energy U, entropy S, temperature T, pressure p, volume V, chemical potential μ and number of particles N of a thermodynamic system is $dU=TdS-pdV+\mu dN$. That U is an exact differential implies that

(a) $-\left.\frac{\partial p}{\partial S}\right|_{V,N}=\left.\frac{\partial T}{\partial V}\right|_{S,N}$

(b) $p\left.\frac{\partial U}{\partial T}\right|_{S,N}=S\left.\frac{\partial U}{\partial V}\right|_{S,\mu}$

(c) $p\left.\frac{\partial U}{\partial T}\right|_{S,N}=-\frac{1}{T}\left.\frac{\partial U}{\partial V}\right|_{S,\mu}$

(d) $\left.\frac{\partial p}{\partial S}\right|_{V,N}=\left.\frac{\partial T}{\partial V}\right|_{S,N}$

Ans. : (a)

Solution: $df = Adx + Bdy + Cdz$

If f is perfect differential then, $A = T, B = -p,\ C = \mu$

$x = S\ y = V,\ z = N$

$$\left(\frac{\partial A}{\partial y}\right)_{x,z} = \left(\frac{\partial B}{\partial x}\right)_{y,z}$$

$$\left(\frac{\partial A}{\partial z}\right)_{x,y} = \left(\frac{\partial C}{\partial x}\right)_{y,z}, \left(\frac{\partial B}{\partial z}\right)_{x,y} = \left(\frac{\partial C}{\partial y}\right)_{x,z}$$

$$\left(\frac{\partial T}{\partial V}\right)_{S,N} = -\left(\frac{\partial P}{\partial S}\right)_{V,N}$$

Q82. The number of microstates of a gas of N particles in a volume V and of internal energy U, is given by

$$\Omega(U,V,N) = (V - Nb)^N \left(\frac{aU}{N}\right)^{3N/2}$$

(where a and b are positive constants). Its pressure P, volume V and temperature T, are related by

(a) $\left(P + \frac{aN}{V}\right)(V - Nb) = Nk_BT$ (b) $\left(P - \frac{aN}{V^2}\right)(V - Nb) = Nk_BT$

(c) $PV = Nk_BT$ (d) $P(V - Nb) = Nk_BT$

Ans. : (d)

Solution: $\Omega(u,v,N) = (V - Nb)^N \left(\frac{aU}{N}\right)^{\frac{3N}{2}}$

$$S = k\ln\Omega = Nk\left[\ln(V - Nb) + \frac{3}{2}\ln\left(\frac{aU}{N}\right)\right]$$

$$dS = \frac{1}{T}dU + \frac{P}{T}dV$$

$$\left(\frac{\partial S}{\partial U}\right)_V = \frac{1}{T} \Rightarrow U = \frac{3}{2}NkT$$

$$\left(\frac{\partial S}{\partial V}\right)_U = \frac{P}{T} \Rightarrow \frac{P}{T} = \frac{Nk}{(V - Nb)}$$

$$P(V - Nb) = NkT$$

Q83. A closed system having three non-degenerate energy levels with energies $E = 0, \pm \in$, is at temperature T. For $\in = 2k_B T$, the probability of finding the system in the state with energy $E = 0$, is

(a) $\frac{1}{(1+2\cosh 2)}$ (b) $\frac{1}{(2\cosh 2)}$ (c) $\frac{1}{2}\cosh 2$ (d) $\frac{1}{\cosh 2}$

Ans. : (a)

Solution: Partition function is $z = 1 + \left(e^{\frac{-\in}{kT}} + e^{\frac{\in}{kT}}\right) \Rightarrow z = 1 + 2\left(\frac{e^{\frac{-\in}{kT}} + e^{\frac{\in}{kT}}}{2}\right) = 1 + 2\cosh\frac{\in}{kT}$

Probability that system has energy, $E = 0$

$$P(E=0) = \frac{1}{1+2\cosh\frac{\in}{kT}}$$

put $\in = 2kT$

$$P(E=0) = \frac{1}{1+2\cosh 2}$$

Q84. Two non-degenerate energy levels with energies 0 and $\in$ are occupied by N non-interacting particles at a temperature T. Using classical statistics, the average internal energy of the system is

(a) $\frac{N\in}{(1+e^{\in/k_B T})}$ (b) $\frac{N\in}{(1-e^{\in/k_B T})}$ (c) $N\in e^{-\in/k_B T}$ (d) $\frac{3}{2}Nk_B T$

Ans. : (a)

Solution: For one particle

Quantum mechanical energy is, $\langle U \rangle = \frac{\in \exp\left(-\frac{\in}{kT}\right)}{1+e^{\frac{-\in}{kT}}}$

For N particle, $\langle U \rangle = \frac{N\in \exp\left(-\frac{\in}{kT}\right)}{1+\exp\left(-\frac{\in}{kT}\right)} \Rightarrow \langle U \rangle = \frac{N\in}{1+\exp\left(\frac{\in}{kT}\right)}$

Q85. Consider a quantum system of non-interacting bosons in contact with a particle bath. The probability of finding no particle in a given single particle quantum state is 10^{-6}. The average number of particles in that state is of the order of

(a) 10^3 (b) 10^6 (c) 10^9 (d) 10^{12}

Ans. : (b)

NET/JRF (JUNE-2018)

Q86. Which of the following statements concerning the coefficient of volume expansion α and the isothermal compressibility κ of a solid is true?

(a) α and κ are both intensive variables

(b) α is an intensive and κ is an extensive variable

(c) α is an extensive and κ is an intensive variable

(d) α and κ are both extensive variables

Ans. : (a)

Solution: $\alpha = \frac{1}{V}\left(\frac{dV}{dT}\right)$, $\kappa = -\frac{1}{V}\left(\frac{\partial P}{\partial V}\right)_T$ both are intensive property

Q87. The number of ways of distributing 11 indistinguishable bosons in 3 different energy levels is

(a) 3^{11} (b) 11^3 (c) $\frac{(13)!}{2!(11)!}$ (d) $\frac{(11)!}{3!8!}$

Ans. : (c)

Solution: $n = 11 \quad g = 3$

$$w = \frac{\lfloor n+g-1}{\lfloor n \lfloor g-1} = \frac{\lfloor 11+3-1}{\lfloor 11 \lfloor 2} = \frac{\lfloor 13}{\lfloor 11 \lfloor 2}$$

Q88. The van der Waals equation for one mole of a gas is $\left(p+\frac{a}{V^2}\right)(V-b)=RT$. The corresponding equation of state for n moles of this gas at pressure P, volume V and temperature T, is

(a) $\left(P+\frac{an^2}{V^2}\right)(V-nb)=nRT$ (b) $\left(P+\frac{a}{V^2}\right)(V-nb)=nRT$

(c) $\left(P+\frac{an^2}{V^2}\right)(V-nb)=nRT$ (d) $\left(P+\frac{a}{V^2}\right)(V-nb)=nRT$

Ans. : (a)

Solution: For n mole gas van der Wall equation is

$$\left(p+\frac{an^2}{V}\right)(V-nb)=nRT$$

Q89. In a system of N distinguishable particles, each particle can be in one of two states with energies 0 and $-E$, respectively. The mean energy of the system at temperature T is

(a) $-\frac{1}{2}N\left(1+e^{\varepsilon/k_BT}\right)$ (b) $-NE\left(1+e^{\varepsilon/k_BT}\right)$

(c) $-\frac{1}{2}NE$ (d) $-NE\left(1+e^{-\varepsilon/k_BT}\right)$

Ans. : (d)

Solution: For one particle system

$$\langle E\rangle=\frac{0\times e^{\frac{-0}{k_BT}}+(-E)e^{+E/k_BT}}{e^{-0/k_BT}+e^{E/k_BT}}=\frac{-E\ e^{E/k_BT}}{1+e^{E/k_BT}}=\frac{-E}{e^{-E/k_BT}+1}\langle E\rangle=\frac{-NE}{1+e^{-E/k_BT}}$$

Q90. The pressure P of a system of N particles contained in a volume V at a temperature T is given by $P=nk_BT-\frac{1}{2}an^2+\frac{1}{6}bn^3$, where n is the number density and a and b are temperature independent constants. If the system exhibits a gas-liquid transition, the critical temperature is

(a) $\frac{a}{bk_B}$ (b) $\frac{a}{2b^2k_B}$ (c) $\frac{a^2}{2bk_B}$ (d) $\frac{a^2}{b^2k_B}$

Ans. : (c)

Solution: $P=nk_BT-\frac{1}{2}an^2+\frac{1}{6}bn^3 \quad n=\frac{N}{V}$

For critical condition $\frac{\partial P}{\partial V}=0$ and $\frac{\partial^2 P}{\partial V^2}=0$

$$P=\frac{N}{V}k_BT-\frac{1}{2}a\frac{N^2}{V^2}+\frac{1}{6}b\frac{N^3}{V^3}$$

$$\frac{\partial P}{\partial V}=0\Rightarrow Nk_BT=\frac{aN^2}{V}-\frac{bN^3}{2V^2} \qquad \text{(i)}$$

$$\frac{\partial^2 P}{\partial V^2} = 0 \Rightarrow 2Nk_B T = \frac{3aN^2}{V} - \frac{2bN^3}{V^2} \qquad \text{(ii)}$$

From equation (i) and (ii)

$$V_C = \frac{bN}{a}$$

put the value of $V_C = \frac{b}{a}N$ in equation (i)

$$T = \frac{a^2}{2k_B b}$$

Q91. Consider a particle diffusing in a liquid contained in a large box. The diffusion constant of the particle in the liquid is $1.0\times10^{-2}\,cm^2/s$. The minimum time after which the root-mean-squared displacement becomes more than $6\ cm$ is

(a) 10 min (b) 6 min (c) 30 min (d) $\sqrt{6}$ min

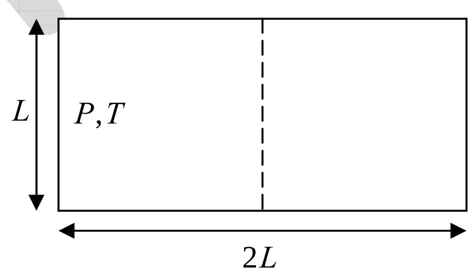
Ans. : (a)

Solution: $\langle r^2 \rangle = 6Dt$

$$\langle r^2 \rangle = (r.m.s)^2 = (6\,c.m)^2 \qquad D = 1\times10^{-2}\,c.m^2/\text{sec}$$

$$t = \frac{\langle r^2 \rangle}{60} = \frac{(6)^2}{6\times1\times10^{-2}} = 600\ \text{sec} = 10\ \text{min}$$

Q92. A thermally insulated chamber of dimensions $(L, L, 2L)$ is partitioned in the middle. One side of the chamber is filled with n moles of an ideal gas at a pressure P and temperature T, while the other side is empty. At $t = 0$, the partition is removed and the gas is allowed to expand freely. The time to reach equilibrium varies as



(a) $n^{1/3}L^{-1}T^{1/2}$ (b) $n^{2/3}LT^{-1/2}$ (c) $n^0 LT^{-1/2}$ (d) $n L^{-1}T^{1/2}$

Ans. : (c)

Solution: From kinetic theory of gases $\langle F\rangle = \frac{m\langle v^2\rangle}{L} = MLt^{-2} = \frac{m}{L}\frac{3kT}{M}$ where $nM = m$ M is molecular mass

$t^{-2} = L^{-2}T \Rightarrow t = LT^{-1/2}$

Q93. The maximum intensity of solar radiation is at the wavelength of $\lambda_{sun} \sim 5000\,\overset{o}{A}$ and corresponds to its surface temperature $T_{sun} \sim 10^4\,K$. If the wavelength of the maximum intensity of an X-ray star is $5\,\overset{o}{A}$, its surface temperature is of the order of

(a) $10^{16}\,K$ (b) $10^{14}\,K$ (c) $10^{10}\,K$ (d) $10^7\,K$

Ans. : (d)

Solution: From Vein's law, $T_{\max}\lambda_{sun} = \text{constant}$

$5000\,A_0^0 \times 10^4 = 5\,\overset{0}{A} \times T$

$T = \frac{5000\times 10^4}{5}$ $\quad T = 10^7\,K$

NET/JRF (DEC - 2018)

Q94. The heat capacity C_V at constant volume of a metal, as a function of temperature, is $\alpha T + \beta T^3$, where α and β are constants. The temperature dependence of the entropy at constant volume is

(a) $\alpha T + \frac{1}{3}\beta T^3$ (b) $\alpha T + \beta T^3$

(c) $\frac{1}{2}\alpha T + \frac{1}{3}\beta T^3$ (d) $\frac{1}{2}\alpha T + \frac{1}{4}\beta T^3$

Ans. : (a)

Solution: $C_V = \alpha T + \beta T^3$

$dS = \frac{dQ}{T} = \frac{C_V\,dT}{T}$

$\int dS = \int\left(\alpha + \beta T^2\right)dT$

$S = \alpha T + \frac{1}{3}\beta T^3$

Q95. The rotational energy levels of a molecule are $E_\ell = \frac{\hbar^2}{2I_0}\ell(\ell+1)$, where $\ell = 0,1,2,...$ and I_0 is its moment of inertia. The contribution of the rotational motion to the Helmholtz free energy per molecule, at low temperatures in a dilute gas of these molecules, is approximately

(a) $-k_BT\left(1+\frac{\hbar^2}{I_0k_BT}\right)$ (b) $-k_BTe^{-\frac{\hbar^2}{I_0k_BT}}$

(c) $-k_BT$ (d) $-3k_BTe^{-\frac{\hbar^2}{I_0k_BT}}$

Ans. : (d)

Solution: $E_\ell = \frac{\hbar^2}{2I_0}\ell(\ell+1) \qquad \ell = 0,1,2,...$

$$z = \sum_{\ell=0}^{\infty}(2\ell+1)e^{\frac{-\beta\hbar^2\ell(\ell+1)}{2I_0}}$$

$$z = 1+\sum_{\ell=0}^{\infty}(2\ell+1)\,e^{\frac{-\hbar^2\ell(\ell+1)}{2I_0k_BT}}$$

$$F = -k_BT\ln z = -k_BT\ln\left(1+\sum_{\ell=1}^{\infty}(2\ell+1)\,e^{\frac{-\hbar^2\ell(\ell+1)}{2I_0k_BT}}\right)$$

$$\ln(1+x) = x-\frac{x^2}{2}+...$$

For low temperature, higher temperature can be neglected

$$F = -k_BT\sum_{\ell=1}^{\infty}(2\ell+1)\,e^{-\frac{-\hbar^2\ell(\ell+1)}{2I_0k_BT}} = -k_BT\left[3\,e^{\frac{-\hbar^2}{I_0k_BT}}+...\right] = -3k_BT\,e^{-\frac{\hbar^2}{I_0k_BT}}$$

Q96. The vibrational motion of a diatomic molecule may be considered to be that of a simple harmonic oscillator with angular frequency ω. If a gas of these molecules is at temperature T, what is the probability that a randomly picked molecule will be found in its lowest vibrational state?

(a) $1-e^{-\frac{\hbar\omega}{k_BT}}$ (b) $e^{-\frac{\hbar\omega}{2k_BT}}$ (c) $\tanh\left(\frac{\hbar\omega}{k_BT}\right)$ (d) $\frac{1}{2}\text{cosec}\,h\left(\frac{\hbar\omega}{2k_BT}\right)$

Ans. : (a)

Solution: $E = \left(n + \frac{1}{2}\right)\hbar\omega \qquad n = 0, 1, 2, ..$

$$z = e^{\frac{-\hbar\omega}{2k_BT}} + e^{\frac{-3\hbar\omega}{2k_BT}} + e^{\frac{-5\hbar\omega}{-k_BT}}$$

$$P(G.S.) = \frac{e^{\frac{-\hbar\omega}{2k_BT}}}{e^{\frac{-\hbar\omega}{2k_BT}} + e^{\frac{-3\hbar\omega}{2k_BT}} + ...} = \frac{e^{\frac{-\hbar\omega}{2k_BT}}}{e^{\frac{-\hbar\omega}{2k_BT}}\left(1 + e^{\frac{-\hbar\omega}{k_BT}} + ...\right)} = \frac{1}{\frac{1}{1 - e^{\frac{-\hbar\omega}{k_BT}}}} = 1 - e^{\frac{-\hbar\omega}{k_BT}}$$

Q97. Consider an ideal Fermi gas in a grand canonical ensemble at a constant chemical potential. The variance of the occupation number of the single particle energy level with mean occupation number $\bar{n}$ is

(a) $\bar{n}(1 - \bar{n})$ (b) $\sqrt{\bar{n}}$ (c) $\bar{n}$ (d) $\frac{1}{\sqrt{\bar{n}}}$

Ans. : (a)

Solution: $\bar{n} = k_BT\frac{1}{z}\left(\frac{\partial z}{\partial \mu}\right)_{V,T} = \frac{1}{e^{(\in-\mu)/k_BT} + 1}$

Variance $= k_BT\left(\frac{d\bar{n}}{d\mu}\right)_{V,T} = \bar{n}(1 - \bar{n})$

Note: This may also be divided using simple Bernoulli distribution.

Q98. The Hamiltonian of a one-dimensional Ising model of N spins (N large) is

$$H = -J\sum_{i=1}^{N}\sigma_i\sigma_{i+1}$$

where the spin $\sigma_i = \pm 1$ and J is a positive constant. At inverse temperature $\beta = \frac{1}{k_BT}$, the correlation function between the nearest neighbor spins $(\sigma_i\sigma_{i+1})$ is

(a) $\frac{e^{-\beta J}}{\left(e^{\beta J} + e^{-\beta J}\right)}$ (b) $e^{-2\beta J}$

(c) $\tanh(\beta J)$ (d) $\coth(\beta J)$

Ans. : (c)

Solution: $\langle \sigma_i \cdot \sigma_{i+1} \rangle = \dfrac{\sum \sigma_i \cdot \sigma_{i+1}}{N-1} = \dfrac{\sum \sigma_i \cdot \sigma_{i+1}}{N} N >> 1 = \dfrac{1-1}{-JN}$ (i)

For such an Ising model for $N >> 1$

$$z = (\cosh \beta J)^N$$

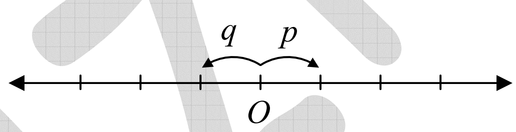
Average Energy $= \dfrac{-\partial}{\partial \beta} \ln z$

$$= -N \frac{1}{\cosh \beta J} \sinh \beta J \cdot J$$

$$= -NJ \tanh \beta J \quad \text{(ii)}$$

$$\langle \sigma_i \cdot \sigma_{i+1} \rangle = \frac{-Nj \tanh \beta j}{-jN} = \tanh \beta j$$

Q99. A particle hops on a one-dimensional lattice with lattice spacing a. The probability of the particle to hop to the neighboring site to its right is p, while the corresponding probability to hop to the left is $q = 1 - p$. The root-mean squared deviation $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$ in displacement after N steps, is



(a) $a\sqrt{Npq}$ (b) $aN\sqrt{pq}$ (c) $2a\sqrt{Npq}$ (d) $a\sqrt{N}$

Ans. : (c)

Solution: The standard deviation of Binomial distribution $= \sqrt{Npq}$

Step size $= 2a$ (L & R)

Mean square displacement $= 2a\sqrt{Npq}$

Q100. The energy levels accessible to a molecule have energies $E_1 = 0$, $E_2 = \Delta$ and $E_3 = 2\Delta$ (where Δ is a constant). A gas of these molecules is in thermal equilibrium at temperature T. The specific heat at constant volume in the high temperature limit $(k_B T \gg \Delta)$ varies with temperature as

(a) $\dfrac{1}{T^{3/2}}$ (b) $\dfrac{1}{T^3}$ (c) $\dfrac{1}{T}$ (d) $\dfrac{1}{T^2}$

Ans. : (d)

Solution: $z = e^{0} + e^{-\Delta/k_BT} + e^{-2\Delta/k_BT}$ $\quad \dfrac{\Delta}{k_BT} << 1$

$$z = 1 + e^{-\Delta/k_BT} + e^{-2\Delta/k_BT}$$

$$A = -k_BT\ln z = -k_BT\ln\left[1 + e^{-\Delta/k_BT} + e^{-2\Delta/k_BT}\right]$$

$$A = -k_BT\ln\left[1 + 1 - \frac{\Delta}{k_BT}.... + 1 - \frac{2\Delta}{k_BT}...\right]$$

$$= -k_BT\ln\left[3 - \frac{3\Delta}{k_BT}\right]$$

$$\frac{\partial\Delta}{\partial T} = -k_B\left[1\ln\left[3 - \frac{3\Delta}{k_BT}\right] + T\cdot\frac{1}{3 - \frac{3\Delta}{k_BT}}\cdot\frac{3\Delta}{k_BT^2}\right]$$

$$\frac{\partial^2\Delta}{\partial T^2} = -k_B\left[\frac{1}{3 - \frac{3\Delta}{k_BT}}\frac{3\Delta}{k_BT^2} + \frac{1}{3 - \frac{3\Delta}{k_BT}}\cdot\frac{-3}{k_BT^2} + \frac{3\Delta}{k_BT}(-1)\left(3 - \frac{3\Delta}{k_BT}\right)^{-2}\frac{3\Delta}{k_BT^2}\right]$$

$$= \frac{k_B\Delta^2}{T\left(k_BT - \Delta\right)^2} = \frac{k_B\Delta^2}{T\cdot k_B^2T^2}$$

$$C_V = -T\frac{\partial^2 A}{\partial T^2} = \alpha\frac{1}{T^2}$$