Some results on K-frames

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Abstract. In this paper we present some results on K-frames when $K \in B(H)$ is an injective closed range operator. Also we give a condition on K-frames $\{f_n\}_{n\in N}$ and $\{g_n\}_{n\in N}$ so that $\{f_n + g_n\}_{n\in N}$ is again a K-frame for H. Finally, Schatten class operators are also discussed in terms of K-frames.

Keywords: K-frames, Schatten class operators.

1. Introduction

Frames in Hilbert spaces were introduced by R.J. Duffin and A.C. Schaffer. Later Daubechies, Grossmann and Meyer gave a strong place to frames in harmonic analysis. Frame theory plays an important role in signal processing, sampling theory, coding and communications and so on. Frames were introduced as a better replacement to orthonormal basis. We refer [2] for an introduction to frame theory.

K-frames were introduced by L. Gavruta, to study atomic systems with respect to bounded linear operators. K-frames are more general than classical frames. In K-frames the lower bound only holds for the elements in the range of K.

Some basic definitions and results related to frames and K-frames are contained in section 2. In section 3 we have included some new results on K-frames. Section 4 contains our main results relating K-frames and operators in Schatten classes.

Throughout this paper, H is a separable Hilbert space and we denote by $B(H)$, the space of all linear bounded operators on H. For $K \in B(H)$, we denote $R(K)$ the range of K. Also, $GL(H)$ denote the set of all bounded linear operators which have bounded inverses.

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2. Preliminaries

For a separable Hilbert space H, a sequence $\{f_n\}_{n\in N} \subset H$ is said to be a **frame** ([2]) for H if there exist $A, B > 0$ such that

$$A\|x\|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, f_n\rangle|^2 \leq B\|x\|^2,$$

for all $x \in H$.If $A = B$, we say that $\{f_n\}_{n\in N}$ is a tight frame in H. Let $K \in B(H)$. We say that $\{f_n\}_{n\in N} \subset H$ is a K**-frame** ([3]) for H if there exist constants $A, B > 0$ such that

$$A\|K^*x\|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, f_n\rangle|^2 \leq B\|x\|^2,$$

for all $x \in H$.

If $\{f_n\}_{n\in N} \subset H$ is an ordinary frame for H, then $\{Kf_n\}_{n\in N}$ is a K-frame for H ([5]). If $T \in B(H)$ and $\{f_n\}_{n\in N}$ is a K-frame for H, then $\{Tf_n\}_{n\in N}$ is a TK-frame for H ([5]). If $\{f_n\}_{n\in N} \subset H$ is a K-frame for H, then $\{K^N f_n\}_{n\in N}$ is a K^N-frame for H where $N \geq 1$ is a fixed integer ([5]). $\mathcal{F}_K(H) \subset \mathcal{F}_M(H)$ if and only if $R(K) \supset R(M)$ where $\mathcal{F}_K(H), \mathcal{F}_M(H)$ denote the set of all K-frames and M-frames on H ([4]). Also, we use the result: $T \in B(H)$ is an injective and closed range operator if and only if there exists a constant $c > 0$ such that $c\|x\|^2 \leq \|Tx\|^2$, for all $x \in H$ ([6]),in the proof of our main results.

3. K-frames

In this section we present our results on K-frames.

Theorem 3.1. *Let $K \in B(H)$ be an injective and closed range operator.If $\{f_n\}_{n\in N}$ is a frame for $R(K)$, then $\{K^*f_n\}_{n\in N}$ is a frame for H and hence $\{KK^*f_n\}_{n\in N}$ is a K-frame for H.*

Proof. Let $\{f_n\}_{n\in N}$ be a frame for $R(K)$.Then there exist constants $A, B > 0$ such that, for all $x \in R(K)$,

$$A\|x\|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, f_n\rangle|^2 \leq B\|x\|^2.$$

Also,by our assumption, there exists $c > 0$ such that $c\|x\|^2 \leq \|Kx\|^2$, for all $x \in H$. For $x \in H, Kx \in R(K)$, and we get

$$A\|Kx\|^2 \leq \sum_{n=1}^{n=\infty} |\langle Kx, f_n\rangle|^2 \leq B\|Kx\|^2.$$

Therefore,

$$Ac\|x\|^2 \leq A\|Kx\|^2 \leq \sum_{n=1}^{n=\infty} |\langle Kx, f_n\rangle|^2 \leq B\|Kx\|^2 \leq B\alpha^2\|x\|^2,$$

for all $x \in H$ and for some $\alpha > 0$, i.e.

$$E\|x\|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, K^* f_n\rangle|^2 \leq F\|x\|^2,$$

for all $x \in H$ where $E = Ac > 0, F = B\alpha^2 > 0$. Therefore, $\{K^* f_n\}_{n\in N}$ is a frame for H and hence $\{KK^* f_n\}_{n\in N}$ is a K-frame for H. □

Corollary 3.2. *Let $K \in B(H)$ be an injective and closed range operator and $\{f_n\}_{n\in N} \subset H$ be such that $\{(K^{-1})^* f_n\}_{n\in N}$ is a frame for $R(K)$.Then $\{f_n\}_{n\in N}$ is a frame for H.*

Theorem 3.3. *Suppose $\{f_n\}_{n\in N}$ is a K-frame for H where K^* is an injective and closed range operator. Then there exist constants $A, B > 0$ such that*

$$A\|K^* x\|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, f_n\rangle|^2 \leq B\|K^* x\|^2,$$

for all $x \in H$.

Proof. Since $\{f_n\}_{n\in N}$ is a K-frame for H, there exist constants $C, D > 0$ such that

$$C\|K^* x\|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, f_n\rangle|^2 \leq D\|x\|^2,$$

for all $x \in H$. Since $K^* \in B(H)$ is an injective and closed range operator, there exist $d > 0$ such that

$$d\|x\|^2 \leq \|K^* x\|^2,$$

for all $x \in H$. Therefore, for all $x \in H$,

$$C\|K^* x\|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, f_n\rangle|^2 \leq D\|x\|^2 \leq (D/d)\|K^* x\|^2,$$

for all $x \in H$ there exist $A = C, B = D/d > 0$ such that

$$A\|K^* x\|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, f_n\rangle|^2 \leq B\|K^* x\|^2.$$

□

Corollary 3.4. *Suppose $\{f_n\}_{n\in N}$ is a K-frame for H where K^* is an injective and closed range operator. Then $\{f_n\}_{n\in N}$ is a frame for H.*

Definition 3.5. *A sequence $\{f_n\}_{n\in N} \subset H$ is said to be a $2K$-frame for H if there exist $A, B > 0$ such that*

$$A\|K^*x\|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, f_n\rangle|^2 \leq B\|K^*x\|^2,$$

for all $x \in H$.

Theorem 3.6. *Let $\{f_n\}_{n\in N}$ be a K-frame for H with bounds A_1, B_1 and $\{g_n\}_{n\in N}$ be a $2K$-frame for H with bounds A_2, B_2 such that $0 < B_2 < A_1$. Then $\{f_n+g_n\}_{n\in N}$ is a K-frame for H with frame bounds A_1-B_2 and $B_1+B_2\|K^*\|^2$.*

Proof. By definition of K-frame and $2K$-frame, we have

$$A_1\|K^*x\|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, f_n\rangle|^2 \leq B_1\|x\|^2$$

and

$$A_2\|K^*x\|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, g_n\rangle|^2 \leq B_2\|K^*x\|^2,$$

for all $x \in H$. Consider,

$$\sum_{n=1}^{n=\infty} |\langle x, f_n+g_n\rangle|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, f_n\rangle|^2 + \sum_{n=1}^{n=\infty} |\langle x, g_n\rangle|^2 \tag{1}$$

$$\leq B_1\|x\|^2 + B_2\|K^*x\|^2 \tag{2}$$

$$\leq (B_1 + B_2\|K^*\|^2)\|x\|^2, \tag{3}$$

for all $x \in H$.

Consider,

$$\sum_{n=1}^{n=\infty} |\langle x, f_n\rangle|^2 = \sum_{n=1}^{n=\infty} |\langle x, f_n + g_n - g_n\rangle|^2 \tag{4}$$

$$\leq \sum_{n=1}^{n=\infty} |\langle x, f_n+g_n\rangle|^2 + \sum_{n=1}^{n=\infty} |\langle x, g_n\rangle|^2. \tag{5}$$

This implies that,

$$A_1\|K^*x\|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, f_n+g_n\rangle|^2 + B_2\|K^*x\|^2$$

$$i.e. \sum_{n=1}^{n=\infty} |\langle x, f_n+g_n\rangle|^2 \geq (A_1 - B_2)\|K^*x\|^2$$

where $A_1 - B_2 > 0$.This completes the proof. $\square$

4. K-frames and operators in Schatten classes

Definition 4.1 ([7]). *Let T be a compact operator on a separable Hilbert space H. Given $0 < p < \infty$, we define the* ***Schatten*** *p* ***- class*** *of H, denoted by $S_p(H)$ or simply S_p, to be the space of all compact operators T on H with its singular value sequence $\{\lambda_n\}$ belonging to l^p.$S_p(H)$ is a two sided ideal in $B(H)$.*

Following two theorems by H. Bingyang, L.H. Khoi and K. Zhu gives a characterization for Schatten p-class operators in terms of frames.

Theorem 4.2 ([1]). *Suppose T is a compact operator on H and $2 \le p < \infty$.Then the following conditions are equivalent:*

(*a*) $T \in S_p$;

(*b*) $\{\|Te_n\|\}_{n\in N} \in l^p$ *for every orthonormal basis* $\{e_n\}_{n\in N}$ *in* H;

(*c*) $\{\|Tf_n\|\}_{n\in N} \in l^p$ *for every frame* $\{f_n\}_{n\in N}$ *in* H.

Theorem 4.3 ([1]). *Suppose T is a compact operator on H and $0 \le p \le 2$.Then the following conditions are equivalent:*

(*a*) $T \in S_p$;

(*b*) $\{\|Te_n\|\}_{n\in N} \in l^p$ *for some orthonormal basis* $\{e_n\}_{n\in N}$ *in* H;

(*c*) $\{\|Tf_n\|\}_{n\in N} \in l^p$ *for some frame* $\{f_n\}_{n\in N}$ *in* H.

At first we focus on the case where $2 \le p < \infty$.

Theorem 4.4. *Suppose T is a compact operator on H and $K \in B(H)$. If T is in the Schatten class S_p, then $\{\|Tf_n\|\}_{n\in N} \in l^p$ for every K-frame $\{f_n\}_{n\in N}$ in H, where $2 \le p < \infty$.*

Proof. Suppose $T \in S_p$,$2 \le p < \infty$.

Let $\{f_n\}_{n\in N}$ be a K-frame for H and $\{e_n\}_{n\in N}$ be an orthonormal basis for H. Then $\{h_n\}_{n\in N} = \{f_n\}_{n\in N} \bigcup \{e_n\}_{n\in N}$ is a frame for H and $\{\|Th_n\|\}_{n\in N} \in l^p$,$2 \le p < \infty$. Therefore $\{\|Tf_n\|\}_{n\in N} \in l^p$,$2 \le p < \infty$ and the result is proved. □

Theorem 4.5. *Suppose T is a compact operator on H and $K \in B(H)$.If $\{\|Tf_n\|\}_{n\in N} \in l^p$ for every K-frame $\{f_n\}_{n\in N}$ in H, then $\{\|TKe_n\|\}_{n\in N} \in l^p$ for every orthonormal basis $\{e_n\}_{n\in N}$ in H, where $2 \le p < \infty$.*

Proof. Let $\{e_n\}_{n\in N}$ be an orthonormal basis for H. Then $\{Ke_n\}_{n\in N}$ is a K-frame for H. Therefore by our assumption $\{\|TKe_n\|\}_{n\in N} \in l^p$,$2 \le p < \infty$.Hence $\{\|TKe_n\|\}_{n\in N} \in l^p$ for every orthonormal basis $\{e_n\}_{n\in N}$ in H. □

Theorem 4.6. *Suppose T is a compact operator on H and $K \in GL(H)$ and $2 \le p < \infty$.Then the following are equivalent:*

(*a*) *T is in the Schatten class S_p;*

(*b*) $\{\|Tf_n\|\}_{n\in N} \in l^p$ *for every K-frame* $\{f_n\}_{n\in N}$ *in H.*

Proof. Clearly, (a) implies (b) holds by Theorem 4.4. Now suppose (b) holds. Then $\{\|TKe_n\|\}_{n\in N} \in l^p$ for every orthonormal basis $\{e_n\}_{n\in N}$ in H. This implies that $TK \in S_p$. Using the fact that S_p is a two- sided ideal in $B(H)$, $TKK^{-1} \in S_p$, i.e. $T \in S_p$. This completes the proof. □

Now we move onto the case where $0 < p \leq 2$.

Theorem 4.7. *Let T be a compact operator on H and $K \in B(H)$. Suppose* $\{\|Te_n\|\}_{n\in N}\in l^p$ *for some orthonormal basis* $\{e_n\}_{n\in N} \subset H$. *Then* $\{\|Tf_n\|\}_{n\in N} \in l^p$ *for some K-frame* $\{f_n\}_{n\in N}$ *for H, where* $0 < p \leq 2$.

Proof. Suppose $\{\|Te_n\|\}_{n\in N} \in l^p$ for some orthonormal basis $\{e_n\}_{n\in N} \subset H$. Then $T \in S_p$, which implies that $TK \in S_p$ for any $K \in B(H)$. By Theorem 4.3, $\{\|TKe_n\|\}_{n\in N} \in l^p$ for some orthonormal basis $\{e_n\}_{n\in N}$ in H. Now take $f_n = Ke_n$, so that $\{f_n\}_{n\in N}$ is a K-frame for H and hence the theorem holds. □

Theorem 4.8. *Let T be a compact operator on H and $K \in B(H)$,where K^* is an injective closed range operator. If* $\{\|Tf_n\|\}_{n\in N} \in l^p$ *for some K-frame* $\{f_n\}_{n\in N}$ *for H,then $T \in S_p$, where* $0 < p \leq 2$.

Proof. By Corollary 3.4,if K^* is an injective closed range operator,then every K-frame is a frame and then applying Theorem 4.3, we get $T \in S_p$. □

5. Acknowledgement

The first author acknowledges the financial support of University Grants Commission.

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Accepted: 20.12.2018

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Palestine Journal of Mathematics , ISSN 2219-5688

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SCALABILITY AND K-FRAMES

Sithara Ramesan and K. T. Ravindran

Communicated by Harikrishnan Panackal

MSC 2010 Classifications: 42C15, 47A63

Keywords and phrases: frames, K-frames, scalable K-frames.

The first author acknowledges the financial support of University Grants Commission. Also the authors like to thank the Editors and Reviewers for the valuable comments and suggestions that helped to improve this manuscript.

Abstract Recent studies on K-frames show that parseval K-frames can be used to manage data loss in signal communication. So the construction of parseval K-frames is desirable and scaling is the easiest way for this construction. In this paper we deal with K-frames which can be scaled to parseval K-frames and tight K-frames and we term such K-frames as scalable K-frames and A-scalable K-frames respectively. We prove some of the results related to scalable K-frames. Also we give characterization result for scalable K-frames.

1 Introduction

Frames in Hilbert spaces were introduced as a generalization of orthonormal bases, by R. J. Duffin and A. C. Schaffer in 1956. Frames have their own advantages compared to bases. The main advantage is the redundancy of frames. Frames span the whole Hilbert space, but the representation of an element using frames need not be unique. This flexibility makes frames an important tool in different areas of research, in theory and in application. Frame theory plays an important role in signal processing, sampling theory, coding and communications and so on. We refer [7] for an introduction to frame theory.

For different applications in theory and application, some special kinds of frames have been introduced. One such frame is K-frame. The concept of K-frames was introduced by L. Gavruta [6], to study atomic systems with respect to bounded linear operators. K-frames are more general than classical frames. Although the span limit of K-frames is restricted to range of K, this generality of K-frames makes K-frames practically important.

G. Kuttyniok, K. A. Okoudjou, F. Philipp, and E. K. Tuley in [4] introduced Scalable frames. Scalable frames have a wide range of applications. Recent studies on K-frames shows that K-frames can be used to deal with the problem of data loss in signal communication. Parseval K-frames and tight K-frames are mainly used for this purpose. So we are interested in constructions which modify a given K-frame into a parseval K-frame or a tight K-frame. The easiest way to get a K-frame from a given K-frame is by scaling the vectors. So it is desirable to have a characterization of K-frames which can be scaled to parseval K-frames or tight K-frames. We term such K-frames as scalable K-frames and A-scalable K-frames respectively.

Some basic definitions and results related to frames and K-frames are contained in section 2. In section 3 we have proved some lead off results on scalable K-frames. Section 4 contains our main result which characterizes scalable K-frames. Throughout this paper, H represent a complex separable Hilbert space, $B(H)$, the space of all linear bounded operators on H. For $K \in B(H)$, we denote $R(K)$ the range of K and $D(K)$ the domain of K. Also, N denote the finite or countable index set.

2 Preliminaries

In this section we give some basic definitions and results about frames and K-frames. For several generalizations and applications in frame theory, refer [1, 2, 3, 9, 10].

Definition 2.1. [7] For a separable Hilbert space H, a sequence $\{f_n\}_{n\in N} \subset H$ is said to be a

That is, there exist a non-negative bounded diagonal operator D in $l^2(N)$ such that $KK^* = T_F^* D^2 T_F$. Now for the converse, suppose there exist a non-negative bounded diagonal operator D in $l^2(N)$ such that $KK^* = T_F^* D^2 T_F$. This implies that

$$({a_1}^2 + {a_2}^2)\langle f, e_1\rangle e_1 + {a_3}^2\langle f, e_2\rangle e_2 = 2\langle f, e_1\rangle e_1 + \langle f, e_2\rangle e_2$$

and we get ${a_1}^2 + {a_2}^2 = 2$ and ${a_3}^2 = 1$. Then taking $\{a_n\}_{n\in N} = \{a_1, a_2, a_3\} = \{1, 1, 1\}$ we get $\{a_n f_n\}_{n\in N}$ is a scalable K-frame.

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Received: August 13th, 2021
Accepted: November 3rd, 2021

MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
Corrected proof
Available online 27.1.2023

research paper
оригинални научни рад
DOI: 10.57016/MV-7942HXLN

SOME RESULTS ON SCALABLE K-FRAMES

Sithara Ramesan and K. T. Ravindran

Abstract. We investigate the scalability of K-frames and derive a characterization for scalable K-frames. We investigate whether or not a particular K-frame is scalable, as well as the existence and uniqueness of scalings. Using the concept of trace of an operator, we analyse the possible scalings, if a given K-frame is scalable. In $\mathbb{C}^n$, we look at the scalability of K-frames independently.

1. Introduction

Frames in Hilbert spaces were introduced by R. J. Duffin and A. C. Schaffer while working on nonharmonic Fourier series. Later Daubechies, Grossmann and Meyer gave a strong place to frames in harmonic analysis. Frame theory have wide range of applications in signal processing, sampling theory, coding and communications etc. Both orthonormal bases and frames in separable Hilbert spaces can be used to express any vector in the Hilbert space. However, the advantage of frames over orthonormal bases is their redundancy. Some particular types of frames have been suggested in theory for various applications. One such frame is K-frame. Notion of K-frames were introduced by L. Gavruta, to study atomic systems with respect to bounded linear operators. K-frames are more general than frames. The span limit of K-frames is restricted to $R(K)$. Scalability of frames was introduced in [6].

In this paper we study about the scalability of K-frames. Recent studies on K-frames show that Parseval K-frames can be used to manage data loss in signal communication. So the construction of Parseval K-frames is desirable and scaling is the easiest way for this construction. In this paper we deal with K-frames which can be scaled to Parseval K-frames and tight K-frames and we term such K-frames as scalable K-frames and A-scalable K-frames, respectively. We prove some of the results related to scalable K-frames. Also we give characterization result for scalable K-frames.

2020 Mathematics Subject Classification: 42C15, 47A63
Keywords and phrases: Frames; K-frames; scalable K-frames.

This implies $\sum_{j\in M} a_j = 1$. □

THEOREM 5.3. *Let $\{f_j\}_{j\in M}$ be a scalable K-frame in $\mathbb{C}^n$ with $\|f_j\| = 1$ for all j. Then M has a subset N such that $\{f_j\}_{j\in N}$ is scalable and $\{f_j f_j^*\}_{j\in N}$ is linearly independent.*

Proof. Using Theorem 5.2, we get, $\frac{KK^*}{\|K^*\|^2} \in \operatorname{con}\{f_j f_j^*\}_{j\in M}$. From Theorem 2.6 and Theorem 2.7, it follows that there exists a subset $J \subseteq M$ such that $\frac{KK^*}{\|K^*\|^2} \in \operatorname{con}\{f_j f_j^*\}_{j\in J}$. and $\{f_j f_j^*\}_{j\in J}$ is linearly independent. □

ACKNOWLEDGEMENT. The first author acknowledges the financial support of University Grants Commission.

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(received 29.09.2021; in revised form 17.06.2022; available online 27.1.2023)

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Published by Društvo matematičara Srbije
Kneza Mihaila 35/IV, P.O.B. 355, 11000 Belgrade, Serbia
E-mail: drustvomatematicara(at)yahoo.com
Fax: +381 11 3036 819

ISSN 2406-0682 (Online), ISSN 0025-5165 (Print)

Reprint

ISSN 0973-9424

INTERNATIONAL JOURNAL OF MATHEMATICAL SCIENCES AND ENGINEERING APPLICATIONS

(IJMSEA)

ASCENT PUBLICATION
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PUNE, INDIA

www.ascent-journals.com

International J. of Math. Sci. & Engg. Appls. (IJMSEA)
ISSN 0973-9424, Vol. 12 No. II (December, 2018), pp. 35-42

ON THE r-INTEGRALS INVOLVING THE I-FUNCTIONS OF SEVERAL VARIABLES

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Abstract

The object of this paper is to obtain the r-integrals involving the I-functions of several variables. On specializing the parameters similar results can be derived in the case of I-functions of two variables and H functions of r and two-variables which include the result proved by Prasanth and Nambisan [2, p.102].

1. Introduction

Notations used:

$_1(a_j;\alpha_j,A_j)_p$ stands for $(a_1;\alpha_1,A_1),(a_2;\alpha_2,A_2),\cdots,(a_p;\alpha_p,A_p)$.

The generalized Fox's H-function, namely I-function of r-variables introduced by Prathima, Nambisan and Santha Kumari [3, p.38] is defined and represented in the following manner:

Key Words : *I-function of two and several complex variables, Multivariable H-functions.*
2000 AMS Subject Classification : 45 A 05.

$$I[z_1,\cdots,z_r] = I^{0,N:m_1,n_1;\cdots;m_r,n_r}_{P,Q:p_1,q_1;\cdots;p_r,q_r}$$
$$\left[\begin{array}{c|c} z_1 & {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_P:\ {}_1(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{p_1};\cdots;\ {}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r} \\ \vdots & \\ z_r & {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)},B_j)_Q:\ {}_1(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1};\cdots;\ {}_1(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r} \end{array}\right]$$
$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\theta_1(s_1)\cdots\theta_r(s_r)\ \ \phi(s_1,\cdots,s_r)\ z_1^{s_1}\cdots z_r s^r ds_1\cdots ds_r, \tag{1.1}$$
where $\phi(s_1,\cdots,s_r)$ and $\theta_i(s_i), i=1,2,\cdots,r$ are given by

$$\phi(s_1,\cdots,s_r) = \frac{\prod_{j=1}^{N}\Gamma^{A_j}\left(1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}s_i\right)}{\prod_{j=1}^{Q}\Gamma^{B_j}\left(1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}s_i\right)\prod_{j=N+1}^{P}\Gamma^{A_j}\left(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}s_i\right)}, \tag{1.2}$$

$$\theta_i(s_i) = \frac{\prod_{j=1}^{m_i}\Gamma^{D_j^{(i)}}(d_j^{(i)}-\delta_j^{(i)}s_i)\prod_{j=1}^{n_i}\Gamma^{C_j^{(i)}}(1-c_j^{(i)}+\gamma_j^{(i)}s_i)}{\prod_{j=m_i+1}^{q_i}\Gamma^{D_j^{(i)}}(1-d_j^{(i)}+\delta_j^{(i)}s_i)\prod_{j=n_i+1}^{p_r}\Gamma^{C_j^{(i)}}(cj^{(i)}-\gamma_j^{(i)}s_i)}. \tag{1.3}$$

Also $z_i \neq 0$ $(i=1,\cdots,r)$, $\omega=\sqrt{-1}, m_j, n_j, p_j, q_j$ $(j=1,\cdots,r)$, N,P,Q are non-negative integers such that $0\leq N\leq P$, $Q\geq 0$, $0\leq m_j\leq q_j$, $0\leq n_j\leq p_j$ $(j=1,2,\cdots,r)$ (not all zero simultaneously). $\alpha_j^{(i)}$ $(j=1,2,\cdots,P, i=1,2,\cdots,r)$, $\beta_j^{(i)}$ $(j=1,2,\cdots,Q, i=1,2,\cdots,r)$, $\gamma_j^{(i)}$ $(j=1,2,\cdots,p_i, i=1,2,\cdots,r)$ and $\delta_j^{(i)}$ $(j=1,2,\cdots,q_i, i=1,2,\cdots,r)$ are positive numbers. a_j $(j=1,2,\cdots,P)$, b_j $(j=1,2,\cdots,Q)$, $c_j^{(i)}$ $(j=1,2,\cdots,p_i, i=1,2,\cdots,r)$ and $d_j^{(i)}$ $(j=1,2,\cdots,q_i, i=1,2,\cdots,r)$are complex numbers.

The exponents A_j $(j=1,2,\cdots,P)$, B_j $(j=1,2,\cdots,Q)$, $C_j^{(i)}$ $(j=1,2,\cdots,p_i, i=1,2,\cdots,r)$ and $D_j^{(i)}$ $(j=1,2,\cdots,q_i, i=1,2,\cdots,r)$ of various gamma functions may take integer values. The I-functioi of r-variables is analytic if

$$\Psi_i = \sum_{j=1}^{P}A_j\alpha_j^{(i)} - \sum_{j=1}^{Q}B_j\beta_j^{(i)} + \sum_{j=1}^{p_i}C_j^{(i)}\gamma_j^{(i)} - \sum_{j=1}^{q_i}D_j^{(i)}\delta_j^{(i)} \leq 0, i=1,2,\cdots,r.$$

The integral (1.1) converges absolutely if $|arg(z_i)| < \frac{1}{2}\pi\Delta_i, i=1,2,\cdots,r$ where

$$\Delta_i = -\sum_{j=N+1}^{P}A_j\alpha_j^{(i)} - \sum_{j=1}^{Q}B_j\beta_j^{(i)} + \sum_{j=1}^{m_i}D_j^{(i)}\delta_j^{(i)} - \sum_{j=m_i+1}^{q_i}D_j^{(i)}\delta_j^{(i)}$$

Reprint

ISSN 0973-9424

INTERNATIONAL JOURNAL OF MATHEMATICAL SCIENCES AND ENGINEERING APPLICATIONS

(IJMSEA)

www.ascent-journals.com

International J. of Math. Sci. & Engg. Appls. (IJMSEA)
ISSN 0973-9424, Vol. 13 No. II (December, 2019), pp. 47-52

AN EXPANSION FORMULA FOR THE I-FUNCTIONS OF SEVERAL VARIABLES

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Abstract

In this paper an expansion formula for the I-function of several variables has been obtained. Many interesting new results can be obtained by specializing the parameters of the I-functions of several variables.

1. Introduction

Notations and Results used :

$(a)_n$ stands for $a(a+1)\cdots(a+n-1)$

${}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)},A_j)_p$ stands for $(a_1;\alpha_1^{(1)},\cdots,\alpha_1^{(r)},A_1),(a_2;\alpha_2^{(1)},$
$\cdots,\alpha_2^{(r)};A_2),\cdots,(a_p;\alpha_p^{(1)},\cdots,\alpha_p^{(r)};A_p)$

Key Words : *I- function of several variables.*

2000 AMS Subject Classification : 45 A 05.

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}, \quad n \geq 1. \tag{1.1}$$

The generalized Fox's H-function, namely I-function of r-variables introduced by Prathima, Nambisan and Santha Kumari [6, p.38] is defined and represented as:

$$I[z_1, \cdots, z_r] = I^{0,n:m_1,n_1;\cdots;m_r,n_r}_{p,q:p_1,q_1;\cdots;p_r,q_r}$$

$$\left[\begin{array}{c|c} z_1 & {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_p :_1 (c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{p_1};\cdots;(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r} \\ \vdots & \\ z_r & (b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_q :_1 (d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1};\cdots;_1 (d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r} \end{array}\right]$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \theta_1(s_1) \cdots \theta_r(s_r) \phi(s_1, \cdots, s_r) z_1^{s_1} \cdots z_r^{s_r} ds_1 \cdots ds_r, \tag{1.2}$$

where $\phi(s_1, \cdots, s_r)$ and $\theta_i(s_i), i = 1, 2, \cdots, r$ are given by,

$$\varphi(s_1,\cdots,s_r) = \frac{\prod_{j=1}^{n} \Gamma^{A_j}\left(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(i)} s_i\right)}{\prod_{j=1}^{q} \Gamma^{B_j}\left(1 - b_j + \sum_{i=1}^{r} \beta_j^{(i)} s_i\right) \prod_{j=n+1}^{p} \Gamma^{A_j}\left(a_j - \sum_{i=1}^{r} \alpha_j^{(i)} s_i\right)}, \tag{1.3}$$

$$\theta_i(s_i) = \frac{\prod_{j=1}^{m_i} \Gamma^{D_j^{(i)}}(d_j^{(i)} - \delta_j^{(i)} s_i) \prod_{j=1}^{n_i} \Gamma^{C_j^{(i)}}(1 - c_j^{(i)} + \gamma_j^{(i)} s_i)}{\prod_{j=m_i+1}^{q_i} \Gamma^{D_j^{(i)}}(1 - d_j^{(i)} + \delta_j^{(i)} s_i) \prod_{j=n_i+1}^{p_i} \Gamma^{C_j^{(i)}}(c_j^{(i)} - \gamma_j^{(i)} s_i)}, \tag{1.4}$$

Also $z_i \neq 0$ $(i = 1, \cdots, r)$, $\omega = \sqrt{-1}$, m_j, n_j, p_j, q_j $(j = 1, \cdots, r)$, n, p, q are non-negative integers such that $0 \leq n \leq p, q \geq 0, 0 \leq m_j \leq q_j, 0 \leq n_j \leq p_j$ $(j = 1, 2, \cdots, r)$ (not all zero simultaneously).

$\alpha_j^{(i)}$ $(j = 1, 2, \cdots, p, i = 1, 2, \cdots, r)$, $\beta_j^{(i)}$ $(j = 1, 2, \cdots, q, i = 1, 2, \cdots, r)$, $\gamma_j^{(i)}$ $(j = 1, 2, \cdots, p_i, i = 1, 2, \cdots, r)$, and $\delta_j^{(i)}$ $(j = 1, 2, \cdots, q_i, i = 1, 2, \cdots, r)$ are positive numbers.

a_j $(j = 1, 2, \cdots, p)$, b_j $(j = 1, 2, \cdots, q)$, $c_j^{(i)}$ $(j = 1, 2, \cdots, p_i, i = 1, 2, \cdots, r)$, and $d_j^{(i)}$ $(j = 1, 2, \cdots, q_i, i = 1, 2, \cdots, r)$ are complex numbers. The exponents A_j $(j = 1, 2, \cdots, p)$, B_j $(j = 1, 2, \cdots, q)$, $C_j^{(i)}$ $(j = 1, 2, \cdots, p_i, i = 1, 2, \cdots, r)$ and $D_j^{(i)}$ $(j = 1, 2, \cdots, q_i, i = 1, 2, \cdots, r)$ of various gamma functions may take non integer values. The I-functiion of r-variables is analytic if

$$\Psi_i = \sum_{j=1}^{p} A_j \alpha_j^{(i)} - \sum_{j=1}^{q} B_j \beta_j^{(i)} + \sum_{j=1}^{p_i} C_j^{(i)} \gamma_j^{(i)} - \sum_{j=1}^{q_i} D_j^{(i)} \delta_j^{(i)} \leq 0, \quad i = 1, 2, \cdots, r.$$

Reprint ISSN 0973-9424

INTERNATIONAL JOURNAL OF MATHEMATICAL SCIENCES AND ENGINEERING APPLICATIONS

(IJMSEA)

ASCENT PUBLICATION @ ASCENT PUNE, INDIA

www.ascent-journals.com

International J. of Math. Sci. & Engg. Appls. (IJMSEA)
ISSN 0973-9424, Vol. 13 No. I (June, 2019), pp. 77-85

ON LINEAR GENERATING RELATIONS INVOLVING I-FUNCTIONS OF r-VARIABLES

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Abstract

The object of this paper is to derive four linear generating relations involving the I- functions of r-variables. Special cases include the result proved by Lawrynovicz [2].

1. Introduction

Notations and Results used :

$(a)_n$ stands for $a(a+1)\cdots(a+n-1)$

${}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_p$ stands for $(a_1;\alpha_1^{(1)},\cdots,\alpha_1^{(r)};A_1),(a_2;\alpha_2^{(1)},\cdots,\alpha_2^{(r)};A_2),$
$\cdots,(a_p;\alpha_p^{(1)},\cdots,(a+p;\alpha_p^{(1)},\cdots(\alpha_p^{(r)};A_p)$

$$(\alpha)_n=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)},n\geq 1. \tag{1.1}$$

Key Words : *I- function of r-variables, H-function of r-variables.*

2000 AMS Subject Classification : 45 A 05.

$$\frac{\Gamma(1-\alpha-n)}{\Gamma(\alpha)} = \frac{(-1)^n}{(\alpha)_n} \tag{1.2}$$

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n (t)^n}{n!} = (1-t)^{-\alpha} \tag{1.3}$$

$$\sum \infty_{n=0} \frac{(\alpha)_n (-t)^n}{n!} = (1+t)^{-\alpha}. \tag{1.4}$$

The generalized Fox's H-function, namely I-function of r-variables introduced by Prathima, Nambisan and Santha Kumari [3,p.38] is defined and represented as:

$$I[z_1, \cdots, z_r] = I^{0,N:m_1,n_1,\cdots,m_r,n_r}_{P,Q:p_1,q_1,\cdots,p_r,q_r}$$
$$\left[\begin{array}{c|c} z_1 & {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_P :{}_1(c_j^{(1)},\gamma_j^{(1)};c_j^{(1)})_{p_1};\cdots;{}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r} \\ \vdots & \\ z_r & {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q :{}_1(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1};\cdots;{}_1(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r} \end{array}\right]$$
$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r} \theta_1(s_1)\cdots\theta_r(s_r)\phi(s_1,\cdots,s_r)z_1^{s_1}\cdots z_r^{s_r}\,ds_1\cdots ds_r, \tag{1.5}$$

where $\phi(s_1, \cdots, s_r)$ and $\theta_i(s_i), i = 1, 2, \cdots, r$ are given by

$$\varphi(s_1,\cdots,s_r) = \frac{\prod_{j=1}^{N}\Gamma^{A_j}(1-a_j+\sum_{j=1}^{r}\alpha_j^{(i)}s_i)}{\prod_{j=1}^{Q}\Gamma^{B_j}(1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}s_i)\prod_{j=N+1}^{P}\Gamma^{A_j}(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}s_i)} \tag{1.6}$$

$$\theta_i(s_1) = \frac{\prod_{j=1}^{m_i}\Gamma_j^{(i)}(d_j^{(i)}-\delta_j^{(i)}s_i)\prod_{j=1}^{n_i}\Gamma^{C_j^{(i)}}(1-c_j^{(i)}+\gamma_j^{(i)}s_i)}{\prod_{j=m_i+1}^{q_i}\Gamma^{D_j^{(i)}}(1-d_j^{(i)}+\delta_j^{(i)}s_i)\prod_{j=n_i+1}^{p_i}\Gamma^{C_j^{(i)}}(c_j^{(i)}-\gamma_j^{(i)}s_i)} \tag{1.7}$$

Also $z_i \neq 0$ $(i = 1,\cdots,r), \omega = \sqrt{-1}, m_j, n_j, p_j, q_j$ $(j = 1,\cdots,r)$, N, P, Q are non-negative integers such that $0 \leq N \leq P$, $Q \geq -$, $0 \leq m_j \leq q_j$, $0 \leq n_j \leq p_j$ $(j = 1,2,\cdots,r)$ (not all zero simultaneously), $\alpha_j^{(i)}$ $(j = 1,2,\cdots,P, i = 1,2,\cdots,r)$, $\beta_j^{(i)}$ $(j = 1,2,\cdots,Q, i = 1,2,\cdots,r)$, $\gamma_j^{(i)}$ $(j = 1,2,\cdots,p_i, i = 1,2,\cdots,r)$ and $\delta_j^{(i)}$ $(j = 1,2,\cdots,q_i, i = 1,2,\cdots,r)$ are positive numbers. a_j $(j = 1,2,\cdots,P)$, b_j $(j = 1,2,\cdots,Q)$, $c_j^{(i)}$ $(j = 1,2,\cdots,p_i, i = 1,2,\cdots,r)$ and $d_j^{(i)}$ $(j = 1,2,\cdots,q_i; i = 1,2,\cdots,r)$ are complex numbers. The exponents A_j $(j = 1,2,\cdots,P)$, B_j $(j = 1,2,\cdots,Q)$, $C_j^{(i)}$ $(j =$

Reprint ISSN 0973-9424

INTERNATIONAL JOURNAL OF MATHEMATICAL SCIENCES AND ENGINEERING APPLICATIONS

(IJMSEA)

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www.ascent-journals.com

International J. of Math. Sci. & Engg. Appls. (IJMSEA)
ISSN 0973-9424, Vol. 13 No. 1 (June, 2019), pp. 87-94

I- FUNCTIONS AND HEAT CONDUCTION IN A SQUARE PLATE

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Abstract

The object of this paper is to obtain a solution of a heat conduction problem in a square plate by using the help of I-functions of several variables. Special cases include the results proved by Ambika A [1] and S. S. Srivastava and Ritu Srivastava [6 ,p.78-80].

1. Introduction

Notations and Results used :

$(a)_n$ stands for $a(a+1)\cdots(a+n-1)$

$(a_n) = \frac{\Gamma(a+n)}{\Gamma(a)}, m \geq 1$

${}_1(a_j;\alpha_j, A_j)_p$ stands for $(a_1;\alpha_1, A_1), (a_2, \alpha_2, A_2), \cdots, (a_p;\alpha_p, A_p)$.

Key Words : *Multivariable I-functions , Multivariable H-functions and Heat conduction in a square plate.*

2000 AMS Subject Classification : 45 A 05.

The I- function of r-variables introduced by Prathima, Nambisan and Santha Kumari [4,p.38] is defined and represented as:

$$I[z_1,\cdots,z_r] = I^{0,N:m_1,n_1;\cdots;m_r,n_r}_{P,Q:p_1,q_1;\cdots;p_r,q_r}$$
$$\left[\begin{array}{c|c} z_1 & {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_P :_1 (c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{p_1};\cdots;_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r} \\ \vdots & \\ z_r & {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_P :_1 (d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1};\cdots;_1(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r} \end{array}\right] \tag{1.1}$$
$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\theta_1(s_1)\cdots\theta_r(s_r)\phi(s_1,\cdots,s_r)z_1^{s_1}\cdots z_r^{s_r}ds_1\cdots,ds_r.$$

where

$$\phi(s_1,\cdots,s_r) = \frac{\prod_{j=1}^{N}\Gamma^{A_j}(1-a_j+\sum_{j=1}^{r}\alpha_j^{(i)}s_i)}{\prod_{j=1}^{Q}\Gamma^{B_j}(1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}s_i)\prod_{j=N+1}^{P}\Gamma^{A_j}(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}s_i)} \tag{1.2}$$

$$\theta_i(s_1) = \frac{\prod_{j=1}^{m_i}\Gamma^{D_j^{(i)}}(d_j^{(i)}-\delta_j^{(i)}s_i)\prod_{j=1}^{n_i}\Gamma^{C_j^{(i)}}(1-c_j^{(i)}+\gamma_j^{(i)}s_i)}{\prod_{j=m_i+1}^{q_i}\Gamma^{D_j^{(i)}}(1-d_j^{(i)}+\delta_j^{(i)}s_i)\prod_{j=n_i+1}^{p_i}\Gamma^{C_j^{(i)}}(c_j^{(i)}-\gamma_j^{(i)}s_i)} \tag{1.3}$$

Also $z_i \neq 0$ $(i = 1,\cdots,r)$, $\omega = \sqrt{-1}$, m_j, n_j, p_j, q_j $(j = 1,\cdots,r)$, N, P, Q are non-negative integers such that $0 \leq N \leq P$, $Q \geq 0$, $0 \leq m_j \leq q_j$, $0 \leq n_j \leq p_j$ $(j = 1,2,\cdots,r)$ (not all zero simultaneously), $\alpha_j^{(i)}$ $(j = 1,2,\cdots,P, i = 1,2,\cdots,r)$, $\beta_j^{(i)}$ $(j = 1,2,\cdots,Q, i = 1,2,\cdots,r)$, $\gamma_j^{(i)}$ $(j = 1,2,\cdots,p_i, i = 1,2,\cdots,r)$ and $\delta_j^{(i)}$ $(j = 1,2,\cdots,q_i, i = 1,2,\cdots,r)$ are positive numbers. a_j $(j = 1,2,\cdots,P)$, b_j $(j = 1,2,\cdots,Q)$, $c_j^{(i)}$ $(j = 1,2,\cdots,p_i, i = 1,2,\cdots,r)$ and $d_j^{(i)}$ $(j = 1,2,\cdots,q_i; i = 1,2,\cdots,r)$ are complex numbers. The exponents A_j $(j = 1,2,\cdots,P)$, B_j $(j = 1,2,\cdots,Q)$, $C_j^{(i)}$ $(j = 1,2,\cdots,p_i, i = 1,2,\cdots,r)$ and $D_j^{(i)}$ $(j = 1,2,\cdots,q, i = 1,2,\cdots,r)$ of various gamma functions may take non integer values.

The I-function of r-variables is analytic if

$$\Psi_i = \sum_{j=1}^{P}A_j\alpha_j^{(i)} - \sum_{j=1}^{Q}B_j\beta_j^{(i)} + \sum_{j=1}^{p_i}C_j^{(i)}\gamma_j^{(i)} - \sum_{j=1}^{q_i}D_j^{(i)}\delta_j^{(i)} \leq 0, i = 1,2,\cdots,r.$$

Reprint

ISSN 0973-9424

INTERNATIONAL JOURNAL OF MATHEMATICAL SCIENCES AND ENGINEERING APPLICATIONS

(IJMSEA)

www.ascent-journals.com

International J. of Math. Sci. & Engg. Appls. (IJMSEA)
ISSN 0973-9424, Vol. 12 No. II (December, 2018), pp.1-7

AN INTEGRAL INVOLVING THE PRODUCT OF AN INCOMPLETE GAMMA FUNCTION, GENERALIZED STRUVE'S FUNCTION AND I-FUNCTION OF r-VARIABLES

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Abstract

The object of this paper is to evaluate an integral involving the product of an Incomplete Gamma function, Generalized Struve's function and the I-function of several complex variables. On specializing the parameters similar results can be derived in the case of I-functions of two variables and H functions of r-variables, which include the result proved by Shahul Hameed [5, p. 70].

1. Introduction

Notations used:

${}_1(a_j; \alpha_j, A_j)_p$ stands for $(a_1; \alpha_1, A_1), (a_2; \alpha_2, A_2), \cdots, (a_p; \alpha_p, A_p)$.

Key Words : *Incomplete Gamma function, I-function of two and several complex variables, Multivariable H functions, Generalized Struve's function.*

2000 AMS Subject Classification : 45 A 05.

The generalized Fox's H-function, namely I-function of r-variables introduced by Prathima, Nambisan and Santha Kumari [4] is defined and represented in the following manner:

$$
\begin{aligned}
&I[z_1,\cdots,z_r] \\
&I^{0,N:m_1,n_1;\cdots;m_r,n_r}_{P,Q:p_1,q_1;\cdots;p_r,q_r}\left[\begin{array}{c} z_1 \\ \cdots \\ z_r \end{array}\middle|\begin{array}{l} {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_P : {}_1(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{p_1};\cdots\ {}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r} \\ {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q :\ {}_1(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1};\cdots;\ {}_1(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r} \end{array}\right] \\
&= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\theta_1(s_1)\cdots\theta_r(s_r)\phi(s_1,\cdots,s_r)\, z_1^{s_1}\cdots z_r^{s_r}\,ds_1\cdots ds_r,
\end{aligned} \tag{1.1}
$$

where $\phi(s_1,\cdots,s_r)$ and $\theta_i(s_i), i=1,2,\cdots,r$ are given by,

$$
\phi(s_1,\cdots,s_r) = \frac{\prod\limits_{j=1}^{N}\Gamma^{A_j}\left(1-a_j+\sum\limits_{i=1}^{r}\alpha_j^{(i)}s_i\right)}{\prod\limits_{j=1}^{Q}\Gamma^{B_j}\left(1-b_j+\sum\limits_{i=1}^{r}\beta_j^{(i)}s_i\right)\prod\limits_{j=N+1}^{P}\Gamma^{A_j}\left(a_j-\sum\limits_{i=1}^{r}\alpha_j^{(i)}s_i\right)}, \tag{1.2}
$$

$$
\theta_i(s_i) = \frac{\prod\limits_{j=1}^{m_i}\Gamma^{D_j^{(i)}}(d_j^{(i)}-\delta_j^{(i)}s_i)\prod\limits_{j=1}^{n_i}\Gamma^{C_j^{(i)}}(1-c_j^{(i)}+\gamma_j^{(i)}s_i)}{\prod\limits_{j=m_i+1}^{q_i}\Gamma^{D_j^{(i)}}(1-d_j^{(i)}+\delta_j^{(i)}s_i)\prod\limits_{j=n_i+1}^{p_i}\Gamma^{C_j^{(i)}}(c_j^{(i)}-\gamma_j^{(i)}s_i)}.
$$

Also $z_i \neq -\ (i=1,\cdots,r)$, $\omega=\sqrt{-1}, m_j, n_j, p_j, q_j\ (j=1,\cdots,r)$, N,P,Q are non-negative integers such that $0\leq N\leq P$, $Q\geq 0$, $0\leq m_j\leq q_j$, $0\leq n_j\leq p_j$ $(j=1,2,\cdots,r)$ (not all zero simultaneously). $\alpha_j^{(i)}$ $(j=1,2,\cdots,P, i=1,2,\cdots,r)$, $\beta_j^{(i)}(j=1,2,\cdots,Q,i=1,2,\cdots,r)$, $\gamma_j^{(i)}$ $(j=1,2,\cdots,p_i,i=1,2,\cdots,r)$ and $\delta_j^{(i)}$ $(j=1,2,\cdots,q_j,i=1,2,\cdots,r)$ are positive numbers. $a_j\ (j=1,2,\cdots,P), b_j\ (i=1,2,\cdots,Q)$, $c_j^{(i)}$ $(j=1,2,\cdots,p_i,i=1,2,\cdots,r)$ and $d_j^{(i)}$ $(j=1,2,\cdots,q_i,i=1,2,\cdots,r)$ are complex numbers. The exponents A_j $(j=1,2,\cdots,P)$, B_j $(j=1,2,\cdots,Q)$, $C_j^{(i)}$ $(j=1,2,\cdots,p_i,i=1,2,\cdots,r)$ and $D_j^{(i)}$ $(j=1,2,\cdots,q_i,i=1,2,\cdots,r)$ of various gamma functions may take non integer values. The I-function of r variables is analytic if

$$
\Psi_i=\sum_{j=1}^{P}A_j\alpha_j^{(i)}-\sum_{j=1}^{Q}B_j\beta_j^{(i)}+\sum_{j=1}^{p_i}C_j^{(i)}\gamma_j^{(i)}-\sum_{j=1}^{q_i}D_j^{(i)}\delta_j^{(i)}\leq 0, i=1,2,\cdots,r.
$$

The integral (1.1) converges absolutely if $|arg(z_i)|<\frac{1}{2}<\pi\Delta_i, i=1,2,\cdots,r$ where

$$
\begin{aligned}
\Delta_i &= -\sum_{j=n+1}^{P}A_j\alpha_j^{(i)}-\sum_{j=1}^{Q}B_j\beta_j^{(i)}+\sum_{j=1}^{m_i}D_j^{(i)}\delta_j^{(i)}-\sum_{j=m_i+1}^{q_i}D_j^{(i)}\delta_j^{(i)} \\
&\quad +\sum_{j=1}^{n_i}C_j^{(i)}\gamma_j^{(i)}-\sum_{j=n_i+1}^{p_i}C_j^{(i)}\gamma_j^{(i)}>0.
\end{aligned}
$$

Reprint

ISSN 0973-9424

INTERNATIONAL JOURNAL OF MATHEMATICAL SCIENCES AND ENGINEERING APPLICATIONS

(IJMSEA)

ASCENT PUBLICATION @ ASCENT PUNE, INDIA

www.ascent-journals.com

International J. of Math. Sci. & Engg. Appls. (IJMSEA)
ISSN 0973-9424, Vol. 13 No. II (December, 2019), pp. 37-45

APPLICATIONS OF I-FUNCTIONS OF SEVERAL VARIABLES IN STATISTCAL DISTRIBUTIONS

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Abstract

The aim of this paper is to derive generalized multivariate statistical distributions involving the density function as the I-functions. Special cases include the results given by Mohammed [4,p.164].

1. Introduction

Notations and Results used :

$(a)_n$ stands for $a(a+1)\cdots(a+n-1)$

$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$, $n \geq 1$

${}_1(a_j;\alpha_j, A_j)_p$ stands for $(a_1;\alpha_1, A_1), (a_2;\alpha_2, A_2), \cdots, (a_p;\alpha_p, A_p)$

$n = \frac{\Gamma(n+1)}{\Gamma n}$, $n \geq 1$.

Key Words : *Multivariable I-functions, The probability density function, The cumulative distribution function, Characteristic function.*

Prathima [7, p. 38].

I-function of r-variables is defined and represented as,

$$I[z_1,\cdots,z_r] = I^{0,N:m_1,n_1;\cdots;m_r,n_r}_{P,Q:p_1,q_1;\cdots;p_r,q_r}$$

$$\left[\begin{array}{c|c} z_1 & {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_P :_1 (c_j^{(1)},\gamma_j^{(1)}:C_j^{(1)})_{p_1};\cdots,(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r} \\ \vdots & \\ z_r & (b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q :_1 (d_j^{(1)},\delta_j^{(1)}:D_j^{(1)})_{q_1};\cdots;_1 (d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r} \end{array}\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\theta_1(s_1)\cdots\theta_r(s_r)\phi(s_1,\cdots,s_r)z_1^{s_1}\cdots z_r^{s_r}ds_1\cdots ds_r, \tag{1.1}$$

where $\phi(s_1,\cdots,s_r)$ and $\theta_i(s_i), i=1,2,\cdots,r$ are given by,

$$\phi(s_1,\cdots,s_r) = \frac{\prod\limits_{j=1}^{N}\Gamma^{A_j}\left(1-a_j+\sum\limits_{i=1}^{r}\alpha_j^{(i)}s_i\right)}{\prod\limits_{j=1}^{Q}\Gamma^{B_j}\left(1-b_j+\sum\limits_{i=1}^{r}\beta_j^{(i)}s_i\right)\prod\limits_{j=N+1}^{P}\Gamma^{A_j}\left(a_j-\sum\limits_{i=1}^{r}\alpha_j^{(i)}s_i\right)}, \tag{1.2}$$

$$\theta_i(s_i) = \frac{\prod\limits_{j=1}^{m_i}\Gamma_j^{i}(d_j^{(i)}-\delta_j^{(i)}s_i)\prod\limits_{j=1}^{n_i}\Gamma^{C_j^{(i)}}(1-c_j^{(i)}+\gamma_j^{(i)}s_i)}{\prod\limits_{j=m_i+1}^{q_i}\Gamma^{D_j^{(i)}}(1-d_j^{(i)}+\delta_j^{(i)}s_i)\prod_{j=n_i+1}^{p_i}\Gamma^{C_j^{(i)}}(c_j^{(i)}-\gamma_j^{(i)}s_i)}, \tag{1.3}$$

The I-function of r-variables is analytic if

$$\Psi_i = \sum_{j-1}^{P}A_j\alpha_j^{(i)} - \sum_{j-1}^{Q}B_j\beta_j^{(i)} + \sum_{j-1}^{p_i}C_j^{(i)}\gamma_j^{(i)} - \sum_{j-1}^{q_i}D_j^{(i)}\delta_j^{(i)} \le 0, i=1,2,\cdots,r.$$

The integral (1.1) converges absolutely if $|arg(z_i)| < \frac{1}{2}\Delta_i\pi,\quad i=1,2,\cdots,r,$ where

$$\begin{aligned}\Delta_i \;&=\; \left(-\sum_{j-n+1}^{P}A_j\alpha_j^{(i)} - \sum_{j-1}^{Q}B_j\beta_j^{(i)} + \sum_{j-1}^{m_i}D_j^{(i)}\delta_j^{(i)} - \sum_{j-m_i+1}^{q_i}D_j^{(i)}\delta_j^{(i)}\right. \\ &\left. + \sum_{j=1}^{n_i}C_j^{(i)}\gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i}C_j^{(i)}\gamma_j^{(i)}\gamma_j^{(i)}\right) > 0\end{aligned} \tag{1.4}$$

On taking $D_j^{(i)}=1$ $(j=1,2,\cdots,m_i, i=1,2,\cdots,r)$ in (1.1), then I-function will be